RATIONAL PARAMETRIZATIONS OF MODULI SPACES OF CURVES

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Contents

1. Introduction

Rational parametrizations of an algebraic variety are early themes in the history of Algebraic Geometry, present since its remote origins. As is well

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known a rational parametrization of an algebraic variety X, defined over a field k, is a dominant rational map

$$f : k^n \dashrightarrow X.$$

The variety X is said to be unirational if such a rational map exists and rational if f is invertible. Unirational varieties are of course of a very special type, since they satisfy a very special requirement. The requirement is elementary, but the search for rational parametrizations, or for detecting the rationality of a given X, is often a rather difficult and deep question. For this reason these questions repeatedly played a crucial role in the evolution of Algebraic Geometry and still this cyclic history does not seem finished at all.

Nowadays unirational varieties are appropriately inserted in a wider, more approachable class of algebraic varieties, namely rationally connected varieties. In particular, the recent notion of rational connectedness is responsible for an important change of perspective on this subject, see e.g. [K].

On the other hand, the knowledge on how to afford the (uni)rationality problem for a given X appears, nowadays too, as very much indebted to a geometric, quite ad hoc study of the main series of classical examples. Among them certainly we have moduli spaces of algebraic varieties of various types and features. In particular curves and their related moduli spaces became, in some sense, special actors of this subject.

This paper aims to give an account, both historical and geometric, on the diverse geography of rational parametrizations of moduli spaces related to curves. We have recollected, as well as reconnected, several unirationality or rationality constructions for some of these moduli spaces, trying to respect chronology and geometry. A historical report on the different attempts to realize these parametrizations is unwound along the way. It has been also natural to partially extend the picture to related questions, concerning for instance the Kodaira dimension or the families of uniruled subvarieties of the moduli spaces to be considered. It is clear that an exhaustive account of this type is beyond the size of this work. This implies that we had to make choices omitting many more subjects.

Elementary examples of algebraic varieties having a unirational moduli space are represented by hypersurfaces and complete intersections in $\mathbf{P}^n$ of fixed degree or type. The coefficients of their defining equations provide indeed a rational parametrization of the corresponding moduli space.

In particular, this simple fact singles out the side of unirational moduli spaces, overshadowing the other possibilities. In the case of curves too, this situation is a leit motiv of the history we are going to outline.

A further leit motiv along the paper is the presence of a superabundant family of classical, or less classical, examples. We were fascinated by their beauty and this was a reason, certainly not unique, for insisting on them.

We work over the complex field. Just to fix notations we introduce the main moduli spaces to be considered. The starting point is the *moduli spaces*

$\mathcal{M}_g$ *of curves* of genus $g \geq 2$. Over $\mathcal{M}_g$ we have the *universal Picard variety*

$$p : \mathrm{Pic}_{d,g} \to \mathcal{M}_g$$

which is the moduli space of pairs (C, L) such that C is a smooth, irreducible genus g curve and $L \in \mathrm{Pic}^d(C)$. Besides $\mathcal{M}_g$ we will deal with the *moduli spaces* $\mathcal{S}_g$ *of spin curves* and with the *Prym moduli spaces* $\mathcal{R}_g$.

All these spaces embed in $\mathrm{Pic}_{d,g}$ as multisections of p, for $d = g-1$ and for $d = 0$ respectively. We recall that $\mathcal{R}_g$ is the moduli space of pairs (C, L) such that L is non trivial and $L^{\otimes 2} \cong \mathcal{O}_C$. The space $\mathcal{R}_g$ is irreducible. On the other hand $\mathcal{S}_g$ is the moduli space of pairs (C, L) such that $L \in \mathrm{Pic}^{g-1}(C)$ is a theta characteristic, that is, $L^{\otimes 2} \cong \omega_C$. The theta-characteristic L is said to be *even* (respectively *odd*) if $h^0(L)$ is even (odd). The space $\mathcal{S}_g$ splits as the disjoint union

$$\mathcal{S}_g^+ \cup \mathcal{S}_g^-,$$

where $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$ are irreducible. They parametrize even (respectively odd) theta characteristics. All the previous moduli spaces naturally interact with the moduli $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g and with the moduli $\mathcal{F}_g$ of K3 surfaces endowed with a genus g polarization. The role played by $\mathcal{A}_g$ and $\mathcal{F}_g$ is crucial in our survey, though we have not extended to them the discussion about rational parametrizations. We can consider, in some sense, all the above mentioned moduli spaces as the classical ones.

The three chapters of this paper are:

2. *Moduli of curves,*
3. *Moduli of spin curves,*
4. *Prym moduli spaces.*

Let us give a summary of them. The starting point is Severi's conjecture that $\mathcal{M}_g$ is (uni)rational for every g, see [S] p.880. The attempts for proving this conjecture strongly influenced the studies on curves and their moduli during the past century. Moreover they constitute very important roots and motivations to contemporary work in this field. For these reasons we closely follow the history stemming from Severi's conjecture, even if this was finally disproven after sixty years. Therefore we discuss results on families of singular plane curves and Severi varieties. Before Mumford, Harris and Eisenbud disproved the conjecture, a way to approach it was to construct unirational families of singular plane curves of genus g with general moduli. We will describe the first attempts in this direction, due to Severi and then to Beniamino Segre.

Many contemporary studies and authors on moduli of curves are directly linked to Segre's work. We refer in particular to subjects like moduli of k-gonal curves for small k and to Segre-Nagata conjecture. So these themes are present in our historical account.

Severi observed that the unirationality of $\mathcal{M}_g$ follows, for $g \leq 10$, from the rationality of the varieties of nodal plane curves of genus $g \leq 10$ and

degree d, where $d = [\frac{2}{3}g] + 2$ is the minimal degree so that a plane curve of genus g and degree d can be general in moduli.

The next attempt is due to Segre. Instead of nodal curves, he tried to use unirational families of plane curves with arbitrary singularities. His methods are very interesting and deep, though the results are on the negative side. They concern questions of the following type. Let

$$\Sigma \subset |\mathcal{O}_{\mathbf{P}^2}(d)|$$

be an integral family of singular plane curves of geometric genus g. Assume that the natural map $\Sigma \to \mathcal{M}_g$ is dominant. Is it possible that

- ◦ Σ *is a linear system?*
- ◦ Σ *is a scroll?*
- ◦ *the points of* $\mathrm{Sing}\ \Gamma$ *are general if* Γ *is general in* Σ*?*

Furthermore let $C \subset S$ be an embedding of a general curve of genus g in a smooth, rational surface S. A question underlying the previous ones is:

- ◦ *when* $\dim |C| > 0$ *and* $|C|$ *is not an isotrivial family?*

We will address this matter in detail along the paper. We point out that, dropping the request that S be rational out, the latter question is just equivalent to:

- ◦ *is* $\mathcal{M}_g$ *uniruled?*

This brings us to a further step in our survey. Following the history, we will describe the advent of K3 surfaces in the study of moduli of curves and Mukai realizations of canonical curves of low genus as linear sections of homogeneous spaces. Beyond K3 surfaces, let S be any smooth surface and let $C \subset S$. Then we will extend our picture to all what is known in the case C has general moduli and $|C|$ is not isotrivial.

The goal of this preparation is clear: families of pairs (C, S) are uniruled, since $\dim |C| > 0$, and often unirational. Building on them, the final part of the chapter is devoted to the promised geometric constructions for parametrizing $\mathcal{M}_g$ in low genus. We describe the known uniruledness results for $g \leq 16$, the diverse unirationality constructions for $g \leq 14$ and the rational connectedness of $\mathcal{M}_{15}$.

Along the way we come to prove the ruledness of various universal Brill-Noether loci and the unirationality of $\mathrm{Pic}_{d,g}$, where $g \leq 9$. Some non exhaustive discussions on the rationality problem for $\mathcal{M}_g$ is also included.

Chapters 3 and 4 are written from a similar point of view and in the same spirit. In the case of $\mathcal{M}_g$ the transition from the uniruledness cases to the cases where $\mathcal{M}_g$ is of general type is still not completed, since the Kodaira dimension is still unknown for $g \in [17, 21]$. Instead, for the moduli spaces of spin curves $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$, the situation has been recently settled.

In chapter 3 we describe all the unirationality and uniruledness constructions for these moduli spaces. Then we describe the transition from the uniruledness to the case where $\mathcal{S}_g^{\pm}$ is of general type.

This appears to be specially interesting in the case of even spin curves. Here the Kodaira dimension is zero in genus 8, while $\mathcal{S}_g^+$ is of general type for $g \geq 9$ and uniruled for $g \leq 7$.

The transition does not admit intermediate cases for the moduli of odd spin curves, where we have the uniruledness for $g \leq 11$ and a variety of general type for $g \geq 12$. Nevertheless we discuss some possible peculiarities of the uniruled variety $\mathcal{S}_{11}^-$.

Behind the previous picture we have again the geometry of K3 surfaces and of their hyperplane sections. The uniruledness results follow, for some moduli of odd spin curves, just because a general curve of geometric genus $g \leq 11$ is a smooth, nodal in genus 10, hyperplane section of a K3 surface: see 2.3 for more details.

In the case of moduli of even spin curves the situation is more delicate and the uniruledness of $\mathcal{S}_g^+$ is related to a special class of K3 surfaces, namely Nikulin K3-surfaces. One can show that, with the exception of genus 6, a general curve of genus $g \leq 7$ is a smooth hyperplane section of a Nikulin surface. This is then crucial to deduce the uniruledness for $g \leq 7$.

In genus 8, hyperplane sections of Nikulin surfaces form a codimension two proper closed set in $\mathcal{S}_8^+$. The beautiful geometry of genus 8 curves, in particular the fact that they are linear sections of the Grassmannian $G(2,6)$, is very much responsible, in some sense, for the moduli space $\mathcal{S}_8^+$ having Kodaira dimension zero.

In chapter 4 we come a little bit more on the side of rationality results. We report on the rationality of $\mathcal{R}_g$, which is known for $g \leq 4$. Here there are different beautiful proofs, due to different authors and involving a lot of classical geometry.

An important role is played here by the study of embeddings $C \subset S$, where C is a curve of genus g with general moduli, S is a surface endowed with quasi étale double cover $\pi : \tilde{S} \to S$ and C is a very ample Cartier divisor. The line bundle defining π restricts to a non trivial L on C such that $L^{\otimes 2}$ is trivial. Therefore π induces a natural map $|C| \to \mathcal{R}_g$.

We discuss different rationality results for $\mathcal{R}_g$, $g = 2,3,4$ and the rationality of Prym moduli spaces of hyperelliptic curves. Then we pass to some unirationality results, focusing in particular on $\mathcal{R}_6$. We describe two different geometric approaches.

On one side we use the beautiful geometry of Enriques surfaces. An Enriques surface$S$ is endowed with an étale double cover $\pi : \tilde{S} \to S$ defined by ω_S. We use Fano polarizations on S, that is, very ample linear systems $|C|$ on S of curves of genus 6. We prove the unirationality of the moduli space of pairs $(S, \mathcal{O}_S(C))$. Then we prove that the family of pairs (S, C) dominates $\mathcal{R}_6$ and deduce its unirationality.

A second method, which takes the flavor of families of nodal plane curves, is the attempt to parametrize $\mathcal{R}_g$ via rational family of nodal conic bundles over $\mathbf{P}^2$. We consider δ-nodal conic bundles $q : T \to \mathbf{P}^2$ satisfying the condition that each fibre of q is a conic of rank ≥ 2 and the general one is

smooth. The discriminant curve of q is then $\Gamma := \{x \in \mathbf{P}^2 \ / \ \mathrm{rank}\ q^*(x) = 2\}$. Let C be the normalization of Γ. C is endowed with an étale double covering $\pi : \tilde{C} \to C$, parametrizing the irreducible components of $q^*(x), x \in \Gamma$.

We consider conic bundles such that π is non trivial. Then we construct explicitely linear systems of δ-nodal conic bundles dominating $\mathcal{R}_g$, $g \leq 6$.

In all the exposition we rely on the bibliography for complete proofs. Some minor novelties are nevertheless present: examples and suggestions, a proof of the unirationality of $\mathcal{R}_g$, $g \leq 6$.

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2. Moduli of curves

2.1. **Origins and a conjecture of Severi.** In May 1915 Severi publishes 'Sulla classificazione delle curve algebriche e sul teorema di esistenza di Riemann' [S]. This paper, as Severi says, summarizes his recent results on the birational classification of algebraic curves. Admittedly, a wider and complete publication on the same subject was postponed to better times, since World War I was rapidly approaching Italy in those days. The paper contains a sentence which is the starting point of a long history:

> "Ritengo probabile che la varietá H sia razionale o quanto meno che sia riferibile ad un' involuzione di gruppi di punti in uno spazio lineare S_{3p-3}; o, in altri termini, che *nell' equazione di una curva piana di genere p (e per esempio dell' ordine $p+1$) i moduli si possano far comparire razionalmente."*

In the above text, H is the moduli space of curves $\mathcal{M}_g$. Severi conjectures that $\mathcal{M}_g$ is probably rational or at least unirational. The idea is that there exists an irreducible family $\mathcal{P}$ of plane curves of equation

$$\sum_{0 \leq i,j \leq d} f_{ij} X^i Y^j = 0,$$

such that:

(1) The general member of $\mathcal{P}$ is birational to a smooth and irreducible projective curve of genus g,
(2) the f_{ij}'s are rational functions of $3g-3$ parameters,
(3) the corresponding natural map $f : \mathcal{P} \to \mathcal{M}_g$ is dominant.

In the same paper it is pointed out that such a family exists for $g \leq 10$. More precisely the author remarks that there exists a rational family

$$\mathcal{P}_{g,d},$$

of plane projective curves of degree d and geometric genus g, which dominates $\mathcal{M}_g$ via the natural map into the moduli. This implies that:

Theorem (Severi) *$\mathcal{M}_g$ is unirational if $g \leq 10$.*

The proof suggested in the paper relies on the irreducibility of $\mathcal{M}_g$ and Brill-Noether theory, two established results for the standards generally accepted at that time. Let us describe it, after some long preliminaries.

For any curve C of genus g we will denote its Brill-Noether loci as follows,

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) \ / \ h^0(L) \geq r+1\}.$$

By the Brill-Noether theory $W^r_d(C)$ is not empty if $\rho(g,r,d) \geq 0$, where $\rho(g,r,d) := g-(r+1)(g-d+r)$ is the Brill-Noether number. Assume C is a *curve with general moduli* . Then it is well known that

$$\dim \ W^r_d(C) \ = \ \max\{-1, \rho(g,r,d)\}$$

and that $W^r_d(C)$ is integral if $\dim \ W^r_d(C) > 0$. Moreover any $L \in W^r_d(C)$ defines a morphism birational onto its image if $r \geq 2$, cfr. [ACGH] ch. V. For $r=2$ we have

$$\rho(g,r,d) \geq 0 \Longleftrightarrow \frac{3}{2}d - 3 \geq g.$$

Hence a curve C with general moduli is birational to a plane curve Γ of degree d, with $\frac{3}{2}d \geq g+3$. As is well known such a curve Γ is nodal.

Definition 2.1. *A curve Γ is nodal if each $x \in \mathrm{Sing}\ \Gamma$ is an ordinary double point. Γ is δ-nodal if it is nodal and δ is the cardinality of* Sing Γ.

Counting linear conditions one expects that a δ-nodal curve Γ of degree d exists as soon as $\dim |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta \geq 0$ that is

$$\binom{d+2}{2} - 3\delta > 0.$$

Assume the latter inequality and that $g \leq \frac{3}{2}d - 3$. Then the genus formula

$$g = \binom{d-1}{2} - \delta$$

implies that

$$g \leq \frac{3}{2}d - 3 \text{ and } \frac{d^2}{3} - \frac{7}{2}d + 2 \leq 0.$$

The next proposition then follows.

Proposition 2.1. *Let Γ be a δ-nodal curve of degree d which is general in moduli and such that* $\dim \ |\mathcal{O}_{\mathbf{P}^2}(d))| - 3\delta \geq 0$. *Then the latter inequalities hold true. In particular they imply $g \leq 10$ and $d \leq 9$.*

Thus the space for a rational parametrization of $\mathcal{M}_g$, suggested by counting linear conditions, drops down to $g \leq 10$ because of Brill-Noether theory
.

We want to discuss more of this situation. Let $\Gamma \subset \mathbf{P}^2$ be an integral δ-nodal curve of geometric genus g. We will assume $d \geq 3$, so that d is uniquely

defined from g and δ. A first problem in the study of families of nodal curves Γ, is to understand the configuration of their nodes. To this purpose we consider the Hilbert scheme of Sing Γ and its open subset parametrizing 0-dimensional smooth schemes of length δ. This will be denoted as

$$Hilb_\delta(\mathbf{P}^2).$$

In what follows Z denotes an element of $Hilb_\delta(\mathbf{P}^2)$ and $\mathcal{I}_Z$ its ideal sheaf.

Definition 2.2. *Assume $\delta \leq \binom{d-1}{2}$. The family of plane curves*

$$\mathcal{N}_{g,\delta} := \{\Gamma \in |\mathcal{O}_{\mathbf{P}^2}(d)| \ / \ \Gamma \ \ \textit{is integral, } \delta\textit{-nodal of geometric genus } g\}$$

is the Severi variety of δ-nodal plane curves of genus g and degree d.

So far $\mathcal{N}_{g,\delta}$ is a locally closed set endowed with the diagram

$$\begin{array}{ccc} \mathcal{N}_{g,\delta} & \xrightarrow{m_{g,\delta}} & \mathcal{M}_g \\ {\scriptstyle h_{g,\delta}}\downarrow & & \\ Hilb_\delta(\mathbf{P}^2) & & \end{array}$$

Here $m_{g,\delta}$ is the map sending Γ to its moduli point in $\mathcal{M}_g$ and $h_{g,\delta}$ is the map sending Γ to the element Sing Γ of $Hilb_\delta(\mathbf{P}^2)$.

Definition 2.3. *$m_{g,\delta}$ and $h_{g,\delta}$ are the natural maps of $\mathcal{N}_{g,\delta}$.*

Consider the universal family $\mathcal{Z} \subset \mathbf{P}^2 \times Hilb_\delta(\mathbf{P}^2)$ of $Hilb_\delta(\mathbf{P}^2)$. Let $\mathcal{I}_\mathcal{Z}$ be its ideal sheaf and let $p_1 : \mathcal{Z} \to \mathbf{P}^2$, $p_2 : \mathcal{Z} \to Hilb_\delta(\mathbf{P}^2)$ be its projection maps. It is clear that we have an open embedding

$$\mathcal{N}_{g,\delta} \subset \mathbb{P}_{g,\delta},$$

in the projective bundle

$$\mathbb{P}_{g,\delta} := \mathbf{P}p_{2*}(p_1^*\mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}_\mathcal{Z}).$$

Moreover the natural projection $\overline{h}_{g,\delta} : \mathbb{P}_{g,\delta} \to Hilb_\delta(\mathbf{P}^2)$ restricts to $h_{g,\delta}$ on $\mathcal{N}_{g,\delta}$. The fibre of $\overline{h}_{g,\delta}$ at Z is $|\mathcal{I}_Z^2(d)|$, possibly a point.

It is important to stress that $\mathcal{N}_{g,\delta}$ could be reducible, or even empty a priori.

However assume that U is an irreducible component of $\mathcal{N}_{g,\delta}$ and let

$$V \subset \mathbb{P}_{g,\delta}$$

be its closure. Then it follows by semicontinuity that

$$\overline{h}_{g,\delta}/V : V \to \overline{h}_{g,\delta}(V)$$

is a projective bundle over an open set of $h_{g,\delta}(V)$. Since the Hilbert scheme $Hilb_\delta(\mathbf{P}^2)$ is rational, we conclude that:

- *If both $h_{g,\delta}/V$ and $m_{g,\delta}/V$ are dominant then $\mathcal{M}_g$ is unirational.*

This principle summarizes the method outlined by Severi for proving the unirationality of $\mathcal{M}_g$, $g \leq 10$. For $g \leq 10$ the method is effective, the result follows via a case by case proof or, alternatively, via suitable algorithms.

Of course good general reasons are known, nowadays, ensuring that $m_{g,\delta}$ and $h_{g,\delta}$ cannot be simultaneously dominant, with finitely many exceptions.

On the other hand the arguments classically in use put in evidence, for every g and δ, some global properties of the structure of $\mathcal{N}_{g,\delta}$ which are implicitely expected but not granted a priori. This is the case for the following:

Key questions or assumptions

- *$\mathcal{N}_{g,\delta}$ is irreducible, in particular not empty.*
- *the tangent map of $h_{g,\delta}$ has generically maximal rank.*

Both these questions appear to be crucial in the study of families of algebraic curves. We do not address here their long and interesting history, if not for quoting some fundamental answers to them. We recall that $\mathcal{N}_{g,\delta}$ has a natural structure of scheme. Let $\Gamma \in \mathcal{N}_{g,\delta}$ and $Z = \mathrm{Sing}\ \Gamma$. The projective completion of the tangent space to $\mathcal{N}_{g,\delta}$ at Γ is just the linear system

$$|\mathcal{I}_Z(d)|.$$

The irreducibility of $\mathcal{N}_{g,\delta}$ was claimed by Severi in his famous Anhang F of [S1]. The first complete proof of this property appeared much later and it is due to J. Harris, [H2]. Actually one has:

Theorem 2.2. *$\mathcal{N}_{g,\delta}$ is integral of codimension δ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$.*

Coming to the second question, we can rephrase it as follows. Since the fibre of $\overline{h}_{g,\delta}$ at Z is $|\mathcal{I}^2_Z(2)|$, the question we are speaking about is whether such a linear system has minimal dimension. This means

$$\dim\ |\mathcal{I}^2_Z(d)| \ = \ max\ \{-1\ ,\ \dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta\},$$

the dimension predicted by the so called postulation. The number 3δ is indeed the number of linear conditions, to be imposed on a linear system of curves on a smooth algebraic surface, for constructing the linear subsystem of singular curves passing through a fixed set N of δ points. Even when

$$\dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| \geq 3\delta$$

it is not granted that the previous equality holds. Consider for example the Severi variety $\mathcal{N}_{1,9}$ of integral 9-nodal plane sextic curves. In this case we have that $\dim |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta = 0$. Hence it follows that the tangent map of

$$h_{1,9} : \mathcal{N}_{1,9} \to Hilb_9(\mathbf{P}^2)$$

is generically of maximal rank if and only if $h_{1,9}$ is dominant. This is false: take a general $N \in Hilb_9(\mathbf{P}^2)$. Then there is a unique element $E \in |\mathcal{I}^2_N(6)|$ as expected. But this is not an element of $\mathcal{N}_{1,9}$ because E is twice the unique plane cubic through N. Hence it is not nodal.

On the other hand let $\Gamma \in h_{1,9}(\mathcal{N}_{1,9})$ and $Z = \mathrm{Sing}\ \Gamma$. Then $|\mathcal{I}^2_Z(6)|$ is a Halphen pencil of plane elliptic curves. This pencil is generated by Γ and by the unique double plane cubic containing Z.

The answer to our second question is however known: the previous example is in fact the unique exception. In other words we have

Theorem 2.3. *Let $Z \in Hilb_\delta(\mathbf{P}^2)$ be general. Then*

$$\dim\ |\mathcal{I}^2_Z(d)|\ =\ \max\ \{-1\ ,\ \dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta\},$$

unless $d = 6$ and $\delta = 9$.

A modern proof of the theorem is due to Arbarello and Cornalba,[AC3], while the same result appears in Terracini [T], cfr. [CC1] section 1. In particular $|\mathcal{I}^2_Z(d)| = \emptyset$ is empty, for a general Z, if and only if $\dim|\mathcal{O}^2_S(d)| - 3\delta < 0$ or $d = 6$ and $\delta = 9$.

Corollary 2.4. *Let* $\dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta \geq 0$. *Then the tangent map of $h_{g,\delta} : \mathcal{N}_{g,\delta} \to Hilb_\delta(\mathbf{P}^2)$ is generically of maximal rank unless $(g, \delta) = (1, 9)$.*

It follows immediately that:

Proposition 2.5. *The next conditions are equivalent if $(g, \delta) \neq (1, 9)$:*

(1) *$h_{g,\delta} : \mathcal{N}_{g,\delta} \to Hilb_\delta(\mathbf{P}^2)$ is dominant,*
(2) $\dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta \geq 0$.

Corollary 2.6. *Assume* $\dim\ |\mathcal{O}_{\mathbf{P}^2}(d)| - 3\delta \geq 0$ *and $(d, \delta) \neq (1, 9)$. Then the Severi variety $\mathcal{N}_{g,\delta}$ is rational.*

Proof. The assumption implies that $h_{g,\delta}$ is dominant. On the other hand $\mathcal{N}_{g,\delta}$ is birational to a projective bundle over $h_{g,\delta}(\mathcal{N}_{g,\delta})$. Since $Hilb_\delta(\mathbf{P}^2)$ is rational, the statement follows. □

Relying on this basis, see also [AS] for further precisions, Severi's unirationality result follows:

Theorem 2.7 (Severi)**.** *$\mathcal{M}_g$ is unirational for $g \leq 10$.*

Proof. By proposition 2.1 both $h_{g,\delta}$ and $m_{g,\delta}$ are dominant at least if:

- *$d = 6$ and $g \leq 6$,*
- *$d = 8$ and $7 \leq g \leq 8$,*
- *$9 \leq g \leq 10$ and $d = 9$.*

In this range $\mathcal{M}_g$ is dominated by $\mathcal{N}_{g,\delta}$, which is rational by corollary 2.6. □

Remark 2.1. The same proof applies to the universal Brill-Noether loci $\mathcal{W}^2_{d,g}$. Hence they are unirational for all the values of (d, g) satisfying the conditions considered in proposition 2.1. This means that $\mathcal{W}^2_{d,g}$ is unirational in the following range $g \leq \frac{3}{2}d - 3 < 11$. More in general the unirationality of $\mathcal{W}^2_{d,g}$ holds true for $g \leq 9$ and $d \geq g + 2$. This follows because then $\mathcal{W}^2_{d,g} \cong \mathrm{Pic}_{d,g}$ and the latter space is unirational for $g \leq 9$, [Ve1]. On the other hand the bound $g \leq 9$ is sharp for the unirationality of $\mathcal{W}^2_{d,g}$,

$d \geq g+2$, because $\mathrm{Pic}_{d,g}$ has non negative Kodaira dimension if $g \geq 10$ and $(d, 2g-2) = 1$, [BFV]. It seems plausible that the unirationality of $\mathcal{W}^2_{d,g}$ always holds true for $g \leq 9$.

It is the aim of this exposition to emphasize down to earth constructions and examples. Therefore we present some concrete rational parametrizations of $\mathcal{M}_g$, $g \leq 10$, lying behind the previous theorem. We use families of linear systems $|\mathcal{I}^2_Z(d)|$ of nodal curves of genus g and *minimal degree* d such that $d \geq \frac{2}{3}g+2$. Here Z is a set of δ points in general position and $\mathcal{I}_Z$ is its ideal sheaf. The set Z moves appropriately along a subvariety of $Hilb_\delta(\mathbf{P}^2)$. We recall that a 0-dimensional subscheme $Z \subset \mathbf{P}^2$ is in *general position* if the restriction map $H^0(\mathcal{O}_{\mathbf{P}^2}(n)) \to H^0(\mathcal{O}_Z(n))$ has maximal rank for each n.

Example 2.1. $g \leq 6$: in this case we can use a *unique* linear system $|\mathcal{I}^2_Z(d)|$ to parametrize $\mathcal{M}_g$. We omit the standard proof of the next statement.

Proposition 2.8. *Let $\delta \geq 4$ and let $Z \subset \mathbf{P}^2$ be a fixed set of δ points in general position. Then, for $g = 10-\delta$, $\mathcal{M}_g$ is dominated by the natural map*

$$m_g : |\mathcal{I}^2_Z(6)| \to \mathcal{M}_g.$$

Let us describe the structure of m_g for g equal to 6 or 5.

Genus six The equation of $|\mathcal{I}^2_Z(6)|$ can be fixed as

$$z_1(l_2l_3l_4)^2 + z_2(l_1l_3l_4)^2 + z_3(l_1l_2l_4)^2 + z_4(l_1l_2l_3)^2 + al_1l_2 + bl_3l_4 + cl_1l_2l_3l_4 = 0$$

where $(l_1 : l_2 : l_3)$ are coordinates on $\mathbf{P}^2$, $l_4 = l_1+l_2+l_3$ and the forms a, b, c are as follows: $a \in \mathbf{C}[l_1^2, l_2^2]$, $b \in \mathbf{C}[l_3^2, l_4^2]$, $c \in \mathbf{C}[l_1, l_2, l_3]$. The base locus of $|\mathcal{I}^2_Z(6)|$ is the set $B = \{l_1l_2 = l_3l_4 = 0\}$. The map m_6 factors as

$$|\mathcal{I}^2_Z(6)| \xrightarrow{\tilde{m}_6} \mathcal{W}^2_{6,6} \xrightarrow{f} \mathcal{M}_6$$

It is well known that $deg\ f = 5$, see for instance [ACGH] p.209. The branch divisor of f is just the Petri divisor in $\mathcal{M}_6$, parametrizing curves C endowed with an $L \in W^1_4(C)$ such that $L^{\otimes 2}$ is special.

It turns out that the symmetric group S_5 acts on $|\mathcal{I}^2_Z(6)|$ as the group of Cremona transformations generated by linear transformations fixing Z and quadratic transformations centered at three of the four points of Z. Then it follows that the degree of m_6 is 120, cfr. [SB] Corollary 5.

The construction is related to Shepherd-Barron's proof of the rationality of $\mathcal{M}_6$, [SB]. It will be revisited later.

Genus five Consider the analogous factorization

$$|\mathcal{I}^2_Z(6)| \xrightarrow{\tilde{m}_5} \mathcal{W}^2_{6,5} \xrightarrow{f} \mathcal{M}_5$$

In this case $\tilde{m}_5$ is generically injective, because the only linear automorphism fixing Z is the identity. Let $S \subset \mathbf{P}^4$ be the anticanonical embedding of the blowing of $\mathbf{P}^2$ along Z. As is well known the strict transform of an element Γ of $|\mathcal{I}^2_Z(6)|$ is the canonical model C of Γ. S is the smooth base locus of a

pencil P of quadrics. Moreover C is the base locus of a net N of quadrics, in particular P is a line in N. It is easy to conclude that $deg\ f \circ \tilde{m}_5$ is the number of pencils $P' \subset N$ projectively equivalent to P. Notice also that P and P' are projectively equivalent if and only if the same is true for the sets of 5 points in $\mathbf{P}^1$: $P \cap \Delta$ and $P' \cap \Delta$, where Δ is the discriminant quintic curve of the net N.

We will return to the following observation: *we have seen that a general curve of genus $g \leq 6$ appears in a fixed linear system on a fixed surface S.*

Example 2.2. $7 \leq g \leq 9$*:* $|\mathcal{I}^2_Z(d)|$ *cannot be fixed and* $\dim |\mathcal{I}_Z(d)| > 0$.

Genus seven We have $d = 7$ for the minimal degree. A general $\Gamma \in |\mathcal{I}^2_Z(7)|$ is a nodal septic with 8 nodes. Still the normalization C of Γ embeds in a Del Pezzo surface S, which is defined by the blow up $\sigma : S \to \mathbf{P}^2$ of Z. The strict transform of Γ is C. Note that $C^2 = 17$ and $h^0(\mathcal{O}_C(C)) = 11$. Hence we obtain $\dim\ |C| = 11 = \dim |\mathcal{I}^2_Z(7)|$ from the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

$|C|$ cannot dominate $\mathcal{M}_7$. On the other hand $\mathcal{N}_{7,7}$ is birational to a $\mathbf{P}^{11}$-bundle $\mathbb{P}$ over $U = h_{7,7}(\mathcal{N}_{7,7})$ of fibre $|\mathcal{I}^2_Z(7)|$ at Z. Notice that

$$\mathcal{N}_{7,7}/PGL(3) \cong \mathbb{P}/PGL(3) \cong \mathcal{W}^2_{7,7}.$$

Let $\mathcal{P}_8 \cong U/PGL(3)$ be the moduli of 8 general points in $\mathbf{P}^2$. With more effort one can show that $\mathbb{P}$ descends to a $\mathbf{P}^{11}$- bundle on $\mathcal{P}_8$. So it follows

Proposition 2.9. $\mathcal{W}^2_{7,7} \cong \mathcal{P}_8 \times \mathbf{P}^{11}$.

For $g = 8, 9$ the situation is similar, we omit further details.

Example 2.3. $g = 10$*:* $|\mathcal{I}^2_Z(d)|$ *is* 0*-dimensional.*

In genus 10 the minimal degree is $d = 9$ and $\delta = 18$. Therefore it follows that $\dim\ |\mathcal{I}^2_Z(9)| = 0$. Hence $\mathcal{N}_{9,18} \cong Hilb_{18}(\mathbf{P}^2)$ and $\mathcal{W}^2_{9,10}$ is unirational. For $d = 10$ the uniruledness of $\mathcal{W}^2_{d,10}$ can be proved. The proof relies on some arguments to be used frequently, so we outline it. Let (C, L) be a pair defining a general $x \in \mathcal{W}^2_{10,10}$. Then the Petri map $\mu : H^0(L) \otimes H^0(\omega_C(-L)) \to H^0(\omega_C)$ is injective and induces an embedding

$$C \subset \mathbf{P}^2 \times \mathbf{P}^1$$

such that $\mathcal{O}_C(1,1) \cong L \otimes \omega_C(-L) \cong \omega_C$. It turns out that C lies in a smooth surface $S \in |\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(4,3)|$: we omit for brevity the not difficult proof of this fact. Note that S is regular and that $\omega_S \cong \mathcal{O}_S(1,1)$. Hence we have $\mathcal{O}_C(C) \cong \mathcal{O}_C$ by adjunction formula. Furthermore we have $\dim |C| = 1$, this follows from the regularity of S and the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_S(C) \to 0.$$

Let $m : |C| \to \mathcal{W}^2_{10,10}$ be the moduli map. If m is constant, a general $D \in |C|$ is a copy of C. Hence there exists a dominant map $C \times |C| \to S$ whose restriction to $C \times \{D\}$ is the identity map $C \to D$: a contradiction

because S is not ruled. Thus $m(|C|)$ is a rational curve through x. This implies that:

Proposition 2.10. *$\mathcal{W}^2_{10,10}$ is uniruled.*

We point out that the Zariski closure $\mathcal{N}_{9,18}$ is not a *scroll* in $|\mathcal{O}_{\mathbf{P}^2}(9)|$. In other words $\mathcal{N}_{10,18}$ is ruled but it is not ruled by a family of linear subspaces of $|\mathcal{O}_{\mathbf{P}^2}(9)|$ of dimension > 0. The same appears to be true for $\mathcal{N}_{10,26}$. This kind of situation is further analyzed in the next section, for instance in proposition 2.11.

2.2. **When a scroll in $|\mathcal{O}_{\mathbf{P}^2}(d)|$ dominates $\mathcal{M}_g$?** The latter section highlights the fact that the unirationality of $\mathcal{M}_g$, $g \leq 10$, is strongly related to the world of rational surfaces. Then it is natural to ask what one can say on the embeddings

$$C \subset S$$

of a curve C with general moduli in a rational surface S and about the linear systems $|\mathcal{O}_S(C)|$. On the other hand an intrinsic limit of the results we have described is due to the use of families of *nodal* plane curves, instead of more general families. In the next proposition an effect of this limit is observed. Let Γ be any integral plane curve of geometric genus g and let $n : C \to \Gamma$ be its normalization.

Definition 2.4. *Γ is linearly rigid if* $\dim |i_*C| = 0$*, for every factorization $n = f \circ i$, such that $i : C \to S$ is an embedding in a smooth, integral surface and $f : S \to \mathbf{P}^2$ is a birational morphism.*

The next proposition makes quite clear that, to go further with the previous methods, one has to use families of singular plane curves Γ such that the orbit of Γ by the Cremona group of $\mathbf{P}^2$ does not contain a nodal curve.

Proposition 2.11. *Let Γ be general of genus $g \geq 13$. Assume that Γ is the strict transform of a nodal curve by a Cremona transformation. Then Γ is linearly rigid.*

Proof. Let $n = f \circ i$ as above. Let $\sigma : \mathbf{P}^2 \to \mathbf{P}^2$ be a birational map such that Γ is the strict transform of a nodal curve Γ' of degree d'. Solving the indeterminacy of $\sigma \circ f$ we have $\sigma \circ f \circ \phi = \psi$, where $\phi : S' \to S$ and $\psi : S' \to \mathbf{P}^2$ are birational morphisms and S' is smooth. The strict transforms of Γ' by ψ and of Γ by $f \circ \phi$ are the same curve C', biregular to i_*C. Note that

$$\phi_*|C'| = |i_*C|.$$

Hence it suffices to show that $\dim |C'| = 0$. Let $Z' = \mathrm{Sing}\ \Gamma'$. Since $\psi_* : |C'| \to |\mathcal{I}^2_{Z'}(d')|$ is injective, we show that $\dim |\mathcal{I}^2_{Z'}(d')| = 0$. Note that Γ' is a general point of a Severi variety $\mathcal{N}_{g,\delta'}$ dominating $\mathcal{M}_g$. This implies as usual that $\binom{d'-1}{2} - \delta' = g \leq \frac{3}{2}d' - 3$. On the other hand the condition $\dim |C'| \geq 1$ implies that $C'^2 = d'^2 - 4\delta' \geq 0$. Then the two inequalities

imply ${d'}^2 - 9d' - 10 \leq 0$, that is, $0 \leq d' \leq 10$. Since C' is general, it follows $g \leq 12$. This contradicts our assumption. $\square$

The use of families of *not nodal* singular curves is the starting point of a second step in the history we are discussing. This is a first negative step with respect to the conjectured unirationality of $\mathcal{M}_g$. It is due to Beniamino Segre.

In 1928 Segre presented a communication to the International Congress of Mathematicians held in Bologna. This is on linear systems of singular plane curves with general moduli. It summarizes 'Sui moduli delle curve algebriche', [Se1], a paper answering the next questions Q1, Q2, Q3.

Consider the Grassmannian $G_{n,d}$ of n-spaces in $|\mathcal{O}_{\mathbf{P}^2}(d)|$, where $n = 0$ is not excluded. If $t \in G_{n,d}$ we will denote by $\mathbb{P}_t$ the corresponding linear system of plane curves. Let $T \subset G_{n,d}$ be an integral variety such that

$$\mathbb{P} = \bigcup_{t \in T} \mathbb{P}_t$$

is a family of integral plane curves Γ of geometric genus g. Then $\mathbb{P}$ is endowed with the usual diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{m_T} & \mathcal{M}_g \\ {\scriptstyle h_T}\downarrow & & \\ Hilb_\delta(\mathbf{P}^2) & & \end{array}$$

By definition $h_T(\Gamma) = \text{Sing } \Gamma$. m_T is the natural map in $\mathcal{M}_g$. The questions considered by Segre are the following, cfr. [Se1] sections 1-4 and 11:

- [Q1] *Does there exist* $\mathbb{P}$ *such that* m_T *is dominant and* T *is a point?*
- [Q2] *Does there exist* $\mathbb{P}$ *such that both* m_T *and* h_T *are dominant?*
- [Q3] *Does there exist* $\mathbb{P}$ *such that* m_T *is dominant and* $\dim\ \mathbb{P}_t > 0$*?*

Remark 2.2. It is clear that:
(1) *a positive answer to* Q_1 *or* Q_2 *implies that* $\mathcal{M}_g$ *is unirational,*
(2) *a positive answer to* Q_3 *implies that* $\mathcal{M}_g$ *is uniruled.*

Let $\mathbb{H} \subset |\mathcal{O}_{\mathbf{P}^r}(d)|$ be a linear system of hypersurfaces of degree d and let Z be its base scheme. $\mathbb{H}$ is said to be complete if $\mathbb{H} = |\mathcal{I}_Z(d)|$. We recall that:

Definition 2.5. *A complete linear system* $\mathbb{H}$ *is regular if the restriction map* $r : H^0(\mathcal{O}_{\mathbf{P}^r}(d)) \to H^0(\mathcal{O}_Z(d))$ *has maximal rank.*

Assume $h^0(\mathcal{I}_Z(d)) > 0$, then $|\mathcal{I}_Z(d)|$ is regular if and only if $h^1(\mathcal{I}_Z(d)) = 0$. In what follows it will be not restrictive to assume that $\mathbb{P}_t$ is complete.

Let Z_t be the base locus of $\mathbb{P}_t$. Up to shrinking T we can assume that the family $\{Z_t, t \in T\}$ is a flat family of 0-dimensional schemes and that $\dim\ \mathbb{P}_t$ is constant. Let $o \in T$ be a general point and let Γ_o be general in $\mathbb{P}_o$. It

follows from Noether's theorem on the reduction of a plane curve to a plane curve with ordinary singularities, that there exist birational morphisms

$$\sigma_o : S_o \to \mathbf{P}^2 \ , \ \psi_o : S_o \to \mathbf{P}^2$$

so that: (i) S_o is smooth, (ii) the strict transform C_o of Γ_o by σ_o is smooth, (iii) ψ_o/C_o is generically injective and $\Gamma'_o := \psi_o(C_o)$ has ordinary singular points. It is standard to show that, up to a finite base change $\pi : T' \to T$, the fourtuple $(C_o, \Gamma_o, \sigma_o, \psi_o)$ moves in an irreducible family $(C_t, \Gamma_t, \sigma_t, \psi_t), t \in T'$, such that $\sigma_t : S_t \to \mathbf{P}^2$, $\psi_t : S_t \to \mathbf{P}^2$, are birational morphisms, C_t is the strict transform by σ_t of a general $\Gamma_t \in \mathbb{P}_t$ and conditions (i), (ii), (iii) considered above for o are satisfied. Note that $|C_t|$ is the strict transform of $|\mathcal{I}_{Z_t}(d)|$ by σ_t. Moreover the elements of the linear system

$$\mathbb{P}'_t := \sigma_{t*}|C_t| \subset |\mathcal{O}_{\mathbf{P}^2}(d')|$$

are curves of degree d' and genus g with ordinary singularities. We have dim $\mathbb{P}_t = \dim |C_o| =$ dim $\mathbb{P}'_t$. Finally, notice also that:

Lemma 2.12. *The following conditions are equivalent:*

(1) $\mathbb{P}_t$ *is regular,*
(2) $h^1(\mathcal{O}_{S_o}(C_o)) = 0$
(3) $\mathbb{P}'_t$ *is regular.*

Due to the previous remarks, we make from now on the following

<u>*Assumption*</u> *A general element of* $\mathbb{P}$ *has ordinary singularities.*

This assumption is clearly not restrictive in order to fully answer questions Q1 and Q3. Instead, to fully answer question Q2 under such an assumption, one has to rely on some well known conjectures on the regularity of linear systems of plane curves. As we will see in a moment, these conjectures go back to Beniamino Segre.

In his paper Segre exhibits the following answers to the previous questions:

- [Q1] *No* $\mathbb{P}$ *exists for* $g \geq 7$,
- [Q2] *No* $\mathbb{P}$ *exists for* $g \geq 37$ *or for* $g \geq 11$ *and* $\mathbb{P}_t$ *regular,*
- [Q3] *No* $\mathbb{P}$ *exists for* $g \geq 37$ *or for* $g \geq 11$ *and* $\mathbb{P}_t$ *regular.*

Segre points out that the negative answer to Q2 and Q3 could be extended to $g \geq 11$ without further assumptions. This is possible, he says, if one relies on an unproved claim of intuitive evidence, [Se1] 6 p. 79. It can be stated as follows:

<u>*Claim*</u> *Let* $p_1 \dots p_s$ *be general points in* $\mathbf{P}^2$ *and* $\nu_1 \dots \nu_s$ *positive integers. Consider the ideal sheaf* $\mathcal{I}_Z$ *of* $Z = \bigcup_{i=1\dots s} Z_i$, *where* Z_i *is Spec* $\mathcal{O}_{\mathbf{P}^2,p_i}/m_i^{\nu_i}$. *Then the linear system* $|\mathcal{I}_Z(d)|$ *is regular.*

This is probably the remote origin of a well known conjecture. Many years later, in 1961, Segre appropriately reformulates this claim:

Conjecture (B. Segre [Se2]). *Let $p_1, \dots, p_s$ and Z be as above. Assume that an element $\Gamma \in |\mathcal{I}_Z(d)|$ is a reduced curve, then $|\mathcal{I}_Z(d)|$ is regular.*

See [Ci1] for an account on the influence of this conjecture and the so many related conjectures and theorems. We refer in particular to Harbourne-Hirschowitz conjecture, cfr. [Ci1] 4.8, and to the following conjecture we have already seen to work when $\nu_1 = \cdots = \nu_s = 2$.

Conjecture *(A. Hirschowitz) Let Z be defined as in the previous claim. If $|\mathcal{I}_Z(d)|$ is regular then a general $\Gamma \in |\mathcal{I}_Z(d)|$ has ordinary singularities, unless the unique element of $|\mathcal{I}_Z(d)|$ is a cubic of multiplicity m and $m = \nu_1 = \cdots = \nu_s$.*

Our subject is influenced by this frame, in particular the next question is of special interest. Let Γ be any plane curve of genus g with general moduli:

- ○ [Q4] *Is Γ linearly rigid as soon as $g \geq 11$?*

Segre's conjecture and his answer to Q3 imply that this is true at least if the singular points of Γ are in general position. So we want to describe more in detail the arguments of Segre for answering Q3 (and Q2).

The answer to Q3

We start with a family as above $\mathbb{P} = \bigcup \mathbb{P}_t,\ t \in T$, of curves of degree d. As already remarked it is not restrictive, for our purpose, to assume that a general curve $\Gamma \in \mathbb{P}$ has ordinary singularities. Also, it will be not restrictive to assume that $\mathbb{P}$ is equal to its orbit by $PGL(3)$. We denote as

$$\overline{\nu} := \nu_1 > \cdots > \nu_s$$

the decreasing sequence of the multiplicities of the δ singular points of Γ. Let $Z_i \subset \mathrm{Sing}\ \Gamma$ be the set of points of fixed multiplicity $\nu_i,\ i = 1 \dots s$. The cardinality of Z_i is constant for a general Γ, it will be denoted as δ_i. Finally, the product of the Hilbert schemes of 0-dimensional subschemes of $\mathbf{P}^2$ of length ν_i contains a non empty open subset parametrizing elements $(Z_1, \dots, Z_s)$ such that Z_i is supported on ν_i distinct points and $Z_i \cap Z_j\ i \neq j$, $i, j = 1 \dots s$. This open set will be denoted as

$$Hilb^o_{\overline{\nu}}(\mathbf{P}^2).$$

Let $(Z_1, \dots, Z_s) \in Hilb^o_{\overline{\nu}}(\mathbf{P}^2)$. Consider $Z = \bigcup_{i=1\dots s} Z_i$ and the blowing up $\sigma : S \to \mathbf{P}^2$ of Z. Clearly $(Z_1, \dots, Z_s)$ defines a triple

$$(S, \mathcal{O}_S(H), \mathcal{O}_S(C))$$

where S is a smooth rational surface and $\mathcal{O}_S(H)$, $\mathcal{O}_S(C)$ are such that

- ○ *$|H|$ defines the blowing up $\sigma : S \to \mathbf{P}^2$ of a set Z of δ distinct points.*
- ○ *$C \sim dH - \sum_{i=1\dots s} \nu_i E_i$, where $E_i = \sigma^* Z_i$ and $Z = \bigcup Z_i$ is a partition.*

Conversely a triple $(S, \mathcal{O}_S(H), \mathcal{O}_S(C))$ immediately defines a point of $Hilb^o_{\overline{\nu}}(\mathbf{P}^2)$ up to the action of $PGL(3)$. Consider the GIT-quotient

$$\mathcal{P}_{\overline{\nu}} := Hilb^o_{\overline{\nu}}(\mathbf{P}^2)/PGL(3).$$

Then $\mathcal{P}_{\overline{\nu}}$ is birational to the moduli space of the triples $(S, \mathcal{O}_S(H), \mathcal{O}_S(C))$ as above. As expected, this moduli space has clearly dimension

$$2s - 8 = 10\chi(\mathcal{O}_S) - 2K_S^2.$$

Actually, $10\chi(\mathcal{O}_S) - 2K_S^2$ is a lower bound for every irreducible component of the moduli space of surfaces with fixed $\chi(\mathcal{O}_S)$ and K_S^2, see [Ca1]. Consider a general $\Gamma \in \mathbb{P}$ and the standard diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{m_T} & \mathcal{M}_g \\ {\scriptstyle h_T}\big\downarrow & & \\ Hilb^o_{\overline{\nu}}(\mathbf{P}^2) & & \end{array}$$

Let $\sigma : S \to \mathbf{P}^2$ be the blowing up of Sing Γ. Then the strict transform C of Γ is smooth and $|C|$ is the strict transform of $\mathbb{P}_t$. In particular, from the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0,$$

we have

$$\dim\ \mathbb{P}_t = \dim\ |C| = h^0(\mathcal{O}_C(C)).$$

Notice also that

$$h^1(\mathcal{O}_C(C)) = h^1(\mathcal{O}_S(C)).$$

Hence $\mathbb{P}_t$ *is not regular if and only if* $\mathcal{O}_C(C))$ *is special.* Furthermore we have

$$\dim\ \mathcal{M}_g\ \leq \dim\ \mathbb{P} - \dim\ PGL(3) \leq \dim\ \mathcal{P}_{\overline{\nu}} + h^0(\mathcal{O}_C(C)).$$

The latter is a key inequality. Furthermore one has

Lemma 2.13.

(1) $h^0(\mathcal{O}_C(C)) \geq 1$ *so that* $C^2 \geq 0$,
(2) $3g + 5 \leq 2\delta + h^0(\mathcal{O}_C(C))$,
(3) $h^1(\mathcal{O}_C(C)) \leq g$.

Proof. (1) follows from the assumption $\dim\ \mathbb{P}_t > 0$. (2) is equivalent to the latter inequality. (3) follows immediately from $h^0(\mathcal{O}_C(C)) \geq 1$. □

Building on the previous inequalities, the hard part of Segre's work is developed essentially in the abstract lattice

$$\mathbb{L} := \text{Pic}\ S = \oplus_{ij}\mathbb{Z}[E_{ij}] \oplus \mathbb{Z}[H], \quad 1 \leq i \leq s \quad , \quad 1 \leq j \leq \delta_i.$$

Here, keeping the previous notations, $\mathcal{O}_S(H) \cong \sigma^*\mathcal{O}_{\mathbf{P}^2}(1)$ and the E_{ij}'s are the irreducible exceptional divisors of σ.

The main point is to consider the group of isometries $G \subset O(\mathbb{L})$ which is generated by reflections $q : \mathbb{L} \to \mathbb{L}$ induced by quadratic transformations centered at three distinct points of Sing Γ.

We discuss the case where $\mathbb{P}_t$ is regular. By Brill-Noether theory and the lemma, a positive answer to question Q3 for $\mathbb{P}$ implies that:

(1) $g \leq \frac{3}{2}(d-2)$,

(2) $h^0(\mathcal{O}_C(C)) \geq 1$
(3) $3g + 5 \leq 2\delta + h^0(\mathcal{O}_C(C))$.

Since $\mathbb{P}_t$ is regular, we have that $h^1(\mathcal{O}_C(C)) = h^1(\mathcal{O}_S(C)) = 0$. Hence (3) is equivalent to $4g + 4 \leq 2\delta + C^2$. Consider the set of linear forms

$$\{h_p : \mathbb{L} \to \mathbb{Z},\ p \in G\}$$

which are defined as follows: $\forall D \in \mathbb{L}, h_p(D) = \langle p(D), H \rangle$, where $\langle\ ,\rangle$ is the intersection product. Segre shows that:

Theorem 2.14. *Assume $h^1(\mathcal{O}_S(C)) = 0$ and that (1), (2), (3) hold true. If $11 \leq \frac{3}{2}(d-2)$ then there exists $p \in G$ such that $h_p([C]) < d$.*

Now assume $11 \leq g$. Since $g \leq \frac{3}{2}(d-2)$, the theorem implies that there exists a birational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ such that $f(C)$ has degree $d' < d$. Globalizing the construction of f, one can show that there exists a family of curves $\mathbb{P}'$ satisfying all the conditions assumed for $\mathbb{P}$, but of degree $d' < d$. The iteration of this argument yelds a contradiction. Then it follows:

Corollary 2.15. *Assume that C has general moduli, that $\mathcal{O}_C(C)$ is non special and that* dim $|\mathcal{O}_S(C)| \geq 1$. *Then $g < 11$.*

If $\mathcal{O}_C(C)$ is special and effective we have $2h^0\mathcal{O}_C(C)) \leq C^2$ by Clifford's theorem. Then the same argument yelds:

Theorem 2.16. *Assume that (1), (2), (3) hold true. If $37 \leq \frac{3}{2}(d-2)$ then there exists $p \in G$ such that $p([C]) < d$.*

Corollary 2.17. *Assume C has general moduli and that* dim $|\mathcal{O}_S(C)| \geq 1$. *Then $g \leq 37$.*

Let us derive some weak conclusions concerning question Q4: *is any plane curve Γ having general moduli and genus $g \geq 11$ linearly rigid?*

Segre's theorem suggests that such a Γ should be linearly rigid. Equivalently, it seems very possible that no smooth curve C, having general moduli and genus $g \geq 11$, embeds in a rational surface S so that dim $|\mathcal{O}_S(C)| \geq 1$ and the map $m : |\mathcal{O}_S(C)| \to \mathcal{M}_g$ is not constant.

Segre's conjecture implies such a property if Γ has ordinary singular points in general position. Indeed the latter assumptions imply the regularity of the linear system $\mathbb{P}_t$ of all curves having at least the same multiplicity of Γ at each point $x \in$ Sing Γ. Then, by corollary 2.15, it follows dim $\mathbb{P}_t = 0$ and this implies that Γ is linearly rigid.

The non existence, for any $g \geq 11$, of families of non linearly rigid curves Γ with general moduli remains however unproved. Perhaps unexpectedly, Segre discovered that, as soon as g grows, $\mathcal{M}_g$ is far from being covered by rational curves $m(P)$, where P is a pencil on a rational surface and $m : P \to \mathcal{M}_g$ is the moduli map.

2.3. **Curves with general moduli and algebraic surfaces.** Due to year 1938 racial laws and to racial prosecution, Beniamino Segre left Italy for Britain. After Second World War he came back and was professor at the University La Sapienza in Rome, on the chair of Higher Geometry. He retired in the year 1973.

Young students following his lectures were exposed to the main problems in Algebraic Geometry. This was certainly one of the effective ways in which the classical circle of ideas could be passed to young generations.

As an example of this passage, it is natural to mention some work which is directly related to problems Q1, Q2, Q3. These questions were indeed reconsidered in 1975 by E. Arbarello in [Ar]. In particular this paper is a connecting point between the past of our story and the forthcoming part, where not only rational surfaces are in use.

Question Q1 is considered in [Ar] for any surface. The result obtained relies on the case of rational surfaces, due to Castelnuovo and completed by Segre, see [C1] and [Se1]. Joining together all these results we have:

Theorem 2.18. *Let P be a linear system of curves of genus g on a smooth, connected surface S. If P dominates $\mathcal{M}_g$ then $g \leq 6$ and S is rational.*

Proof. Up to replacing S by an appropriate birational model, we can assume that a general $C \in P$ is smooth and that P is base point free. We can also assume $g \geq 3$. Since $|C|$ dominates $\mathcal{M}_g$, we have $\dim\ |C| \geq\ \dim\ P \geq 3g-3$. Hence $\mathcal{O}_C(C)$ is non special of degree $C^2 \geq 4g-3$ and adjunction formula yelds $CK_S \leq -2g+1$. Since $|C|$ is base point free, it follows that $|mK_S|$ is empty for $m \geq 1$. Hence S is ruled and birational to $R \times \mathbf{P}^1$, in particular the projection $p : R \times \mathbf{P}^1 \to R$ induces a finite map $p_C : C \to R$. Since the curves of $|C|$ have general moduli, this is impossible unless R is rational. Hence S is rational. Now let $g \geq 10$, then we have $\dim |C| \geq 3g-3 \geq 2g+7$. Moreover, a well known theorem of Castelnuovo on linear systems of curves on a rational surface, [C1] 1.3, implies that then the elements of $|C|$ are hyperelliptic. This contradicts the generality of C and implies the statement for $g \geq 10$. The cases $g = 7, 8, 9$ are excluded by Segre in [Se1] 4. □

The bound $g \leq 6$, offered by the previous theorem, is sharp. Indeed we have already seen in section 2 that, when $g \leq 6$, the space $\mathcal{M}_g$ is dominated by a fixed linear system P of integral plane sextics with $10-g$ double points.

Later, in 1981, Arbarello and Cornalba reconsidered another remarkable result of Segre in the paper 'Footnotes to a Paper by Beniamino Segre', see [AC1] and [Se3]. As summarized in [AC1] after the title, these papers deal with *the number of g^1_k's on a general k-gonal curve, and the unirationality of the Hurwitz Spaces of k-gonal curves.* We reformulate these issues in the vein of our exposition. Let us consider the Hurwitz space

$$\mathcal{H}_{g,k}$$

of the finite covers of degree k of $\mathbf{P}^1$ by curves of genus g. The following is a well known result proved by Fulton in [Fu2]:

Theorem 2.19. *$\mathcal{H}_{k,g}$ is irreducible.*

This implies that the corresponding universal Brill-Noether locus $\mathcal{W}^1_k$ in $\mathrm{Pic}_{k,g}$ is irreducible too. A first question is:

- [Q5] *Is $\mathcal{H}_{g,k}$, and hence $\mathcal{W}^1_k$, unirational or uniruled for some g?*

Moreover consider the forgetful map

$$f : \mathcal{W}^1_k \to \mathcal{M}_g$$

and denote its image by $\mathcal{M}^1_{k,g}$ as usual. A second question is:

- [Q6] *If $f : \mathcal{W}^1_k \to \mathcal{M}^1_{k,g}$ is generically finite, what is its degree?*

Question Q6 makes sense if $\rho(k, g, 1) \leq 0$. Otherwise f is not generically finite. If $\rho(k, g, 1) = 0$ the answer is offered by the degree of the class of $W^1_k(C)$ in $\mathrm{Pic}^k(C)$. Computing it as in [ACGH] 4.4 p. 320, one obtains:

$$deg\ f = \frac{g!}{(g-k+1)!(g-k+2)!}.$$

If $\rho(k, g, 1)$ is negative, namely if $k < [\frac{g+3}{2}]$, then it is proved in [Se3] that f is generically finite. More precisely it is shown that the fibre of f is finite at the moduli point $x \in \mathcal{M}^1_{k,g}$ of a curve C such that $W^1_{k'}(C) = \emptyset$ for $k' < k$.

Let us sketch briefly some geometric motivations behind the proof that f is generically finite. Let $L \in W^1_k(C)$ and $k < [\frac{g+3}{2}]$. Then the Petri map

$$\mu : H^0(L) \otimes H^0(\omega_C(-L)) \to H^0(\omega_C)$$

is not injective. This follows from geometric Riemann-Roch and the count of dimensions. On the other hand it is well known that $Coker\ \mu$ is isomorphic to the tangent space at L to $W^1_k(C)$, [ACGH] prop. 4.2. Since $\mathcal{W}^1_k$ is irreducible, it follows that f is finite if $Coker\ \mu$ is zero dimensional for a general pair (C, L). Hence it suffices to produce a pair (C, L) such that μ is surjective. This is equivalent to say that $\dim\ \mathrm{Ker}\ \ \mu = g - 2k + 2$. Counting dimensions, it is not restrictive to assume that $|L|$ is a base point free pencil. Then the base point free pencil trick, [ACGH] p. 126, implies that $\dim\ \mathrm{Ker}\ \ \mu = h^0(\omega_C(-2L)) = h^1(L^{\otimes 2})$.

Now consider the plane blown up in a point i.e. the Hirzebruch surface $\mathbb{F}_1$. Let $|F|$ be its ruling and E its exceptional line. Segre shows that the condition $h^1(L^{\otimes 2}) = g - 2k + 2$ is satisfied if C is the normalization of a general nodal integral curve of geometric genus g

$$\Gamma \in |kE + mF|,$$

where $m := [\frac{g+k+3}{2}]$ [Se3] 4 p. 542. In other words C is birational to a plane curve Γ' of degree $d = k + m$ such that Sing Γ' consists of finitely many nodes and a unique other ordinary singularity of multiplicity m. This result of Segre can be strengthened. It is proved in [AC1] that:

Theorem 2.20. *Assume $k < [\frac{g+3}{2}]$ then f has degree one. Moreover $W^1_k(C)$ consists of a single element if $L \in W^1_k(C)$ is globally generated.*

In the proof a new modern tool is essential to go beyond the results of classical geometry of the construction. Deformation theory, in particular first order deformations of singular curves, is indeed used systematically to achieve the main steps. The answer [AC1] provides to question Q5 relies on the same methods. It is somehow a kind of surprise:

Theorem 2.21. *Let $k \leq 5$ then $\mathcal{W}^1_k$ is unirational.*

More precisely this statement follows from theorem 5.3 of [AC1]:

Theorem 2.22. *$\mathcal{H}_{k,g}$ is unirational in the following cases:*

(1) $3 \leq k \leq 5$ *and* $g \geq k - 1$,
(2) $k = 6$ *and* $5 \leq g \leq 10$ *or* $g = 12$,
(3) $k = 7$ *and* $g = 7$.

Continuing in our vein, we remark that this theorem can be also viewed as a study of families of nodal curves on a Hirzebruch surface $S = \mathbb{F}_e$. We have

$$\text{Pic } S \cong \mathbf{Z}[E] \oplus \mathbf{Z}[F],$$

where $[E]$ is the class of a minimal section, $E^2 = -e$, F is the fibre of the natural projection $p : S \to \mathbf{P}^1$. Any curve C in S is endowed with the line bundle $L = \mathcal{O}_C(F)$. Clearly L is in $W^1_k(C)$, where $k := \cdot$. Keeping k fixed, we have $m = E \cdot C$ and $C \sim (m + ke)F + kE$.

To parametrize the Hurwitz space $\mathcal{H}_{k,g}$, one can study in S the families

$$\mathcal{N}_{g,\delta}(S)$$

of integral δ-nodal curves of genus $g = p_a(C) - \delta$ and fixed $m = E \cdot C$.

As for Severi varieties of nodal curves in $\mathbf{P}^2$, we have natural maps

$$Hilb_{g,\delta}(S) \xleftarrow{h_{g,\delta}} \mathcal{N}_{g,\delta}(S) \xrightarrow{p_{g,\delta}} \mathcal{H}_{k,g}.$$

By definition $h_{g,\delta}(\Gamma) = \text{Sing } \Gamma$ and $p_{g,\delta}(\Gamma) = p \circ n$, where $n : C \to \Gamma$ is the normalization map. To prove the latter theorem we can simply assume $e = 1$. Then the range of k in its statement is equivalent to the condition that $\dim |C| - 3\delta \geq 0$ and $p_{g,\delta}$ be dominant. Adequate deformation theoretic arguments are then needed to deduce that $h_{g,\delta}$ is dominant and hence to obtain the unirationality results.

A very general open problem, arising from the previous discussion, is about the structure of a finite cover of degree k in the case of algebraic curves and not only. This could give more informations about the unirationality of $\mathcal{W}^1_k$ for $k = 6$ and maybe more, see [Ge] to have an update on the present knowledge on the case of degree 6 finite covers. It is shown in this paper that $\mathcal{H}_{6,g}$ is unirational for $g \leq 28$ and $g = 30, 31, 35, 36, 40, 45$.

The results in [AC1] imply the unirationality of $\mathcal{M}_g$, $g \leq 10$. It is of some chronological interest to note that this paper still appeared before the

radical change due to the 1982 first paper on the Kodaira dimension of $\mathcal{M}_g$ by Harris and Mumford, [HM]. One can read in [AC1]:

'We want to stress that the unirationality of $\mathcal{M}_g$, $g \leq 10$ (and of $\mathcal{M}^1_{g,3}$ of course) is classically known [10]. It has been recently proved by Sernesi [8], using entirely different methods, that $\mathcal{M}_{12}$ is also unirational. The problem of deciding whether $\mathcal{M}_g$ is unirational is, at the moment, open for $g = 11$ or $g > 12$.'

It is the right moment to recall the fundamental results, due to Eisenbud, Harris and Mumford, on the Kodaira dimension of $\mathcal{M}_g$, [HM], [H1] and [EH]:

Theorem 2.23. *$\mathcal{M}_g$ is of general type for $g \geq 24$.*

We have already described some typical, modern evolutions of the problem posed by Severi on the unirationality $\mathcal{M}_g$. We continue describing further evolutions and perspectives, on the side of uniruledness/unirationality of $\mathcal{M}_g$ in low genus. Reading the previous pages, it is quite clear that we are stressing three typical aspects of the modern evolution of our subject:

(1) a first aspect is the study of deformation theory for (C, S), where C is a genus g curve embedded in a smooth surface S. As an application, one can try to understand when a concrete family of pairs (C, S) dominates $\mathcal{M}_g$.

(2) A second aspect is the study of the families of singular curves C of genus g in a given smooth surface S. In particular one would like to describe the family of all δ-nodal curves of genus g having fixed homology class.

(3) A third aspect is the study of families of pairs (C, S) such that C is a smooth curve in S, it has general moduli and $\dim\ |C| > 0$. This is related to the study of uniruledness and unirationality properties of $\mathcal{M}_g$.

A common feature to (1), (2), (3) is that there is no restriction on the type of surface S to be considered. The study of problems (1) and (2) is a central theme since many years and it is due to many authors, see e.g. [Ser2] for the related deformation theory. Also the recent study on Severi varieties of nodal curves on any surface has several sources, among them [DH], [Tan1], [Tan2], [CS], [CC], [Fu1]. We restrict now to problem (3).

In 2007 E. Sernesi has given a partial answer to such a question. Even if these results appear later in the chronology, it is now the moment to describe them.

The starting point can be a rational curve $R \subset \overline{\mathcal{M}}_g$. Taking its normalization we have a morphism

$$m : \mathbf{P}^1 \to \overline{\mathcal{M}}_g.$$

The general idea is to study, in some sense, the normal bundle to the map m. This can give useful informations on the following question:

- *When does R move in a family of rational curves covering $\overline{\mathcal{M}}_g$?*

Of course the main purpose is to understand by a direct analysis of the deformations of R, whether $\overline{\mathcal{M}}_g$ is ruled or not. One can turn to effective applications of these ideas as follows.

A *rational fibration of genus* g is just a relatively minimal morphism

$$f : S \to \mathbf{P}^1$$

such that S is a smooth surface and each fibre is a stable curve of genus g. Since its target space is $\mathbf{P}^1$, such a fibration is called a rational fibration.

We will assume that f is not isotrivial i.e. the associated morphism

$$m_f : \mathbf{P}^1 \to \overline{\mathcal{M}}_g$$

is not constant. In particular this implies, by Arakelov theory, that $h^0(T_S) = 0$ and $h^0(T_{S/\mathbf{P}^1}) = h^1(T_{S/\mathbf{P}^1}) = 0$, [Ser3] 1.4.

Starting from the functor of sheaves$f_* Hom$, one can consider its first derived functor. Denoting it by Ext^1_f, one can then consider the sequence associated to the local-global spectral sequence for Ext_f. This is in fact the exact sequence of sheaves

$$0 \to R^1 f_* T_{S/\mathbf{P}^1} \to Ext^1_f(\Omega^1_{S/\mathbf{P}^1}, \mathcal{O}_S) \to f_* Ext^1_S(\Omega_{S/\mathbf{P}^1}, \mathcal{O}_S) \to 0.$$

The sheaf in the middle is a vector bundle on $\mathbf{P}^1$. Since f has fibres of genus g its rank is $3g-3$, hence we have

$$Ext^1_f(\Omega^1_{S/\mathbf{P}^1}, \mathcal{O}_S) \cong \oplus_{i=1\dots 3g-3} \mathcal{O}_{\mathbf{P}^1}(a_i),$$

cfr. [Ser3] 1.5. Recall that f is said to be *free* rational fibration if $a_i \geq 0$ $i = 1 \dots 3g-3$. We can also say that f *has general moduli* if the exists a dominant rational map $m : \mathbf{P}^1 \times B \to \overline{\mathcal{M}}_g$ such that B is integral and $m/\mathbf{P}^1 \times \{o\} = m_f$ for some $o \in B$. The previous vector bundle contains many informations about the deformations of the map $m_f : \mathbf{P}^1 \to \overline{\mathcal{M}}_g$. In particular one has that

Theorem 2.24. *If $f : S \to \mathbf{P}^1$ has general moduli, then f is free.*

Building on this basis, Sernesi tries to understand when no free rational fibration of genus g exists. The goal is to find the maximal value of g such that $\overline{\mathcal{M}}_g$ is uniruled. The results obtained put in evidence once more the beautiful interplay between the geometry of $\mathcal{M}_g$ and the algebraic surfaces.

Let $C \subset S$ be general of genus $g \geq 3$ and dim $|C| \geq 1$. S is a smooth surface of Kodaira dimension $k(S)$.

Theorem 2.25 (Sernesi)**.** *Under the previous assumptions one has.*

(1) *Let S be of general type, then* dim $|C| \geq 2$ *and* $K_S^2 \geq 3\chi(\mathcal{O}_S) - 10$ *implies* $g \leq 19$.
(2) *Let S be an elliptic surface such that $k(S) \geq 0$, then $g \leq 16$.*
(3) *Let S be a surface such that $k(S) \geq 0$, then*
   - $g \leq 6$ *if* $p_g = 0$,
   - $g \leq 11$ *if* $p_g = 1$,
   - $g \leq 16$ *if* $p_g = 2$ *and* $h^0(\omega_S(-C)) = 0$.

2.4. **Families and rulings of unirational varieties in $\overline{\mathcal{M}}_g$.** Going back to the chronological order of events, we start this section with K3 surfaces endowed with a genus g polarization. In 1983 Mori and Mukai use these surfaces in [MM] to prove the uniruledness of $\mathcal{M}_{11}$. This is a new step in the study of the global geometry of the moduli of curves of low genus.

Furthermore [MM] is a seminal paper for the systematic use of K3 surfaces in the study of curves and their moduli. Since then Mukai, and many other authors, started to investigate the deep and beautiful relations between K3 surfaces and curves of low genus.

This has important consequences in our subject and we will see a few of them later. Now, forgetting about rational surfaces, we want to profit of the K3-geometry and of some other surfaces. We want to discuss in low genus:

(1) the uniruledness of $\mathcal{M}_g$ and of some loci $\mathcal{W}^r_{d,g}$,
(2) ruledness results for the same loci and for $\mathcal{M}_g$.

A *polarized K3 surface of genus* g is a pair $(S, \mathcal{O}_S(C))$ such that: S is a K3 surface, $p_a(C) = g$ and $\mathcal{O}_S(C)$ is very ample and primitive in Pic S.

The uniruledness of $\mathcal{M}_{11}$ is a consequence of the more general result we are now going to state, see [Mu1] and [MM] as well. Let

$$\mathcal{F}_g$$

be the moduli space of polarized K3 surfaces of genus g. $\mathcal{F}_g$ is known to be irreducible. Moreover it is endowed with a projective bundle

$$p_g : P_g \to \mathcal{F}_g,$$

having fibre $|C|$ at the moduli point of $(S, \mathcal{O}_S(C))$. Consider the map

$$m_g : P_g \to \mathcal{M}_g.$$

Counting dimensions we have dim $P_g = 19 + g$. Hence m_g is not dominant for $g \geq 12$. On the other hand one has:

Theorem 2.26. *m_g is dominant for $g \leq 11$ and $g \neq 10$.*

Of course the uniruledness of $\mathcal{M}_g$ follows for the same values of g. Let us sketch a recreative proof of the theorem for $g = 11$, which is based on the Brill-Noether locus $\mathcal{W}^1_{6,11}$ parametrizing 6-gonal curves of genus 11.

$\underline{m_{11} : P_{11} \to \mathcal{M}_{11} \textit{ is dominant}}$

Proof. Let (C, L) be a pair defining a general point of $\mathcal{W}^1_{6,11}$. We have seen that then $|L|$ is a base point free pencil. Let $H := \omega_C(-L)$, then:

Lemma 2.27. *$h^0(H) = 6$ and H is very ample.*

Proof. Since $h^1(H) = h^0(L) = 2$, we have $h^0(H) = 6$. Assume H is not very ample, then $h^0(H(-d)) \geq 5$ for some effective divisor d of degree 2. But then we would have $h^0(L(d)) \geq 3$ and $W^2_8(C) \neq \emptyset$. This is impossible for dimension reasons, since dim $\mathcal{W}^1_{6,11} >$ dim $\mathcal{W}^2_{8,11}$. $\square$

So we can assume that C is embedded in $\mathbf{P}^5$ by $|H|$ as a connected, smooth, linearly normal curve of degree 14 and genus 11. Let $\mathcal{I}_C$ be the ideal sheaf of C, one computes that $h^0(\mathcal{I}_C(2)) \geq 3$.

Let $\mathcal{H}$ be the open set of the Hilbert scheme of C whose elements are curves with the same properties. Note that, for each $D \in \mathcal{H}$, $|\omega_D(-1)|$ is a base point free g^1_6 and that $\mathcal{H}/PGL(6)$ is birational to $\mathcal{W}^1_{6,11}$. Consider the moduli map $m : \mathcal{H} \to \mathcal{M}_{11}$ and the Zariski closure of its image

$$\mathcal{M}^1_{11,6} := \overline{m(\mathcal{H})}.$$

Then $\mathcal{M}^1_{11,6}$ is the Petri divisor in $\mathcal{M}_{11}$ parametrizing 6-gonal curves.

Lemma 2.28. *For a general $D \in \mathcal{H}$ the following conditions are satisfied:*
1) $h^0(\mathcal{I}_D(2)) = 3$, the base locus of $|\mathcal{I}_D(2)|$ is a smooth K3 surface S.
2) $\mathrm{Pic}\ S \cong \mathbb{Z}[F] \oplus \mathbb{Z}[D]$, where $|F|$ is an elliptic pencil and $L \cong \mathcal{O}_C(F)$.

Proof. 1 and 2 are open conditions on $\mathcal{H}$, hence it suffices to produce one $D \in \mathcal{H}$ satisfying them. Fix an elliptic K3 surface S such that:

$$\mathrm{Pic}\ S \cong \mathbb{Z}[H] \oplus \mathbb{Z}[F],$$

where H is a very ample, smooth integral curve of genus 5, $|F|$ is an elliptic pencil and $DF = 6$. By the surjectivity of the period map, S exists. One computes that no $E \in \mathrm{Pic}\ S$ exists such that $E^2 = 0$ and $EH = 3$. Let $S \subset \mathbf{P}^5$ be the embedding of S by $|H|$. Then, by [SD], S is a complete intersection of 3 quadrics. Since H is very ample the same is true for $H+F$. It is easily seen that a general $D \in |H+F|$ is a smooth, linearly normal curve of genus 11 and degree 14, so that $D \in \mathcal{H}$ and $|\mathcal{O}_D(F)|$ is a base point free g^1_6. Then, to prove that D satisfies 1 and 2, it remains to show that $h^0(\mathcal{I}_D(2)) = 3$. This follows from the standard exact sequence

$$0 \to \mathcal{I}_S(2H) \to \mathcal{I}_D(2H) \to \mathcal{O}_S(2H-D) \to 0$$

because $h^0(\mathcal{O}_S(2H-D)) = 0$. Indeed we have $2H - D = H - F$ so that $(H-F)^2 = -4$ and $H(H-F) = 2$. Assume that $L \in |H-F|$, then it follows that $L = L_1 + L_2$, L_1 and L_2 being disjoint lines. This is numerically impossible in $\mathrm{Pic}\ S$, hence $h^0(\mathcal{I}_S(2)) = h^0(\mathcal{I}_D(2)) = 3$. □

We can now conclude our proof: consider the Zariski closure $\mathcal{P}^1_{11,6} \subset \mathcal{P}_{11}$ of the divisor parametrizing pairs (S, C) such that
(1) $C \subset S$ is a smooth, very ample curve of genus 11,
(2) $\mathrm{Pic}\ S \cong \mathbb{Z}[F] \oplus \mathbb{Z}[C]$, where $|F|$ is an elliptic pencil and $F \cdot C = 6$.

By the lemma, the image of $\mathcal{P}^1_{11,6}$ by $m_{11} : \mathcal{P}_{11} \to \mathcal{M}_{11}$ is $\mathcal{M}^1_{11,6}$. Then, since this is a divisor and $\mathcal{P}_{11}$ is irreducible, either $m_{11}(\mathcal{P}_{11}) = \mathcal{M}^1_{11,6}$ or m_{11} is dominant. Assume $m_{11}(\mathcal{P}_{11}) = \mathcal{M}^1_{11,6}$ and consider a pair (S, C) defining a general $x \in \mathcal{P}_{11}$. Then $\mathrm{Pic}\ S \cong \mathbb{Z}[C]$ and $m_{11}(x)$ is general in $\mathcal{M}^1_{11,6}$, hence C is 6-gonal. Let $L \in W^1_6(C)$, a well known theorem implies that $L \cong \mathcal{O}_C(F)$, where $F \subset S$ is a curve such that $F^2 = 0$, [GL]. This is numerically impossible in $\mathrm{Pic}\ S$. Hence m_{11} is dominant. □

Let $g \leq 11$, $g \neq 10$. Due to the geometry of K3 surfaces and Mukai-Mori theorem, the uniruledness of the universal Brill-Noether locus $\mathcal{W}^r_d$ easily follows in many cases. Let $W \subset \mathcal{W}^r_d$ be an irreducible component, we have:

Proposition 2.29. *Let $g \leq 11$ and $g \neq 10$. Assume that W dominates $\mathcal{M}_g$ and parametrizes pairs (C, L) with L special. Then W is uniruled.*

Proof. Let (C, L) be a pair defining a general $x \in W$. We have $C \subset S$, where S is a K3 surface and Pic $S \cong \mathbb{Z}[C]$. Let $Z \in |L|$ and let $\mathcal{I}_Z$ be its ideal sheaf in S. Since Z is a special divisor, the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{I}_Z(C) \to \mathcal{O}_C(C - Z) \to 0$$

implies dim $|\mathcal{I}_Z(C)| \geq 1$. Moreover, one has $h^0(\mathcal{O}_D(Z)) \geq r + 1$ for each $D \in P := |\mathcal{I}_Z(C)|$. Then $(D, \mathcal{O}_D(Z))$ defines a point of W. It is easy to see that the moduli map $P \to W$ is not constant. Hence W is uniruled. $\square$

We recall that a variety X is d-ruled if it is is birational to $Y \times \mathbf{P}^d$ with $d > 0$. A beautiful theorem of Mukai, [Mu2], says that:

Theorem 2.30. *$m_{11} : P_{11} \to \mathcal{M}_{11}$ is birational. Hence $\mathcal{M}_{11}$ is g-ruled.*

Now we describe further uniruledness constructions going up to genus

$$g \leq 16.$$

Actually there are several of these constructions and they often provide stronger results than uniruledness, like for instance unirationality. We will concentrate on this property in the next section. Here we introduce the constructions to be used and prove for them some ruledness properties.

Besides K3 surfaces, *canonical surfaces which are complete intersection* in $\mathbf{P}^n$ or $\mathbf{P}^a \times \mathbf{P}^b$ will be used. The list of them is of course short: in $\mathbf{P}^n$ complete intersections S of type (5), (3,3), (2,4), (2,2,3), (2,2,2,2) are the only possible cases. Let us see two reasons for using these surfaces.

(1) If C of genus g embeds in S and $\mathcal{O}_C(1)$ is special, then dim $|C| \geq 1$. Therefore $|C|$ defines a rational curve through the moduli point of C.

(2) Possibly C is linked to a second curve B in S of genus $p < g$. This fact can be used to parametrize $\mathcal{M}_g$ via a family of curves of lower genus p.

Definition 2.6. *A ruling of X by unirational varieties of dimension d is a dominant rational map $f : X \to Y$ with unirational fibres of dimension $d > 0$.*

Here is a list of examples in low genus which are interesting for us:

Theorem 2.31.

(1) *genus 15: $\mathcal{W}^1_{9,15}$ has a ruling of rational surfaces,*

(2) *genus 14: $\mathcal{W}^1_{8,14}$ is birational to* $\mathrm{Pic}_{14,8} \times \mathbf{P}^{10}$,

(3) *genus 13: $\mathcal{W}^2_{11,13}$ is dominated by* $\mathrm{Pic}_{12,8} \times \mathbf{P}^8$,

(4) *genus 12: $\mathcal{W}^0_{5,12}$ is birational to* $\mathrm{Pic}_{15,9} \times \mathbf{P}^5$,

(5) *genus 11:* $\mathcal{W}^0_{6,11}$ *is birational to* $\mathrm{Pic}_{13,9} \times \mathbf{P}^3$.

Proof. We refer to [BV] and [Ve1] for a complete account of these cases and their proofs. See also Schreyer's paper [Sch] in this Handbook, where the same methods are simplified via an effective use of Computer Algebra.

(1) *Genus 15, 14, and 12*

Let (C, L) be a general pair such that either $g = 14$ and $L \in W^1_8(C)$ or $g = 15$ and $L \in W^1_9(C)$ or $g = 12$ and $L \in W^0_5(C)$. It is easy to see that $\omega_C(-L)$ is very ample and defines an embedding $C \subset \mathbf{P}^6$. We summarize some further properties of (C, L): see [Ve1] section 4 and [BV] section 3.

Let $\mathcal{I}_C$ be the ideal sheaf of C then:

(i) $h^0(\mathcal{I}_C(2)) = 4$ *for* $g = 15$ *and* $h^0(\mathcal{I}_C(2)) = 5$ *for* $g = 14$ *and* $g = 12$.
(ii) C *is contained in a smooth complete intersection* S *of 4 quadrics.*
(iii) $g = 14$*:* C *is linked to a projectively normal curve* B *of genus 8 by a complete intersection of 5 quadrics and* $h^0(\mathcal{I}_B(2)) = 7$.
(iv) $g = 12$*:* C *is linked to a projectively normal curve* B *of genus 9 by a complete intersection of 5 quadrics and* $h^0(\mathcal{I}_B(2)) = 6$.

(1.1) *The case of genus* 14.

In particular B is smooth of degree 14 and $\mathcal{O}_B(1)$ is a line bundle of degree 14. Clearly, all the mentioned properties of the pair $(B, \mathcal{O}_B(1))$ are true for a general smooth, connected curve in the Hilbert scheme of B. This is irreducible and dominates $\mathrm{Pic}_{14,8}$. Let $B' \subset \mathbf{P}^6$ a general curve of degree 14 and genus 8 such that $(B', \mathcal{O}_{B'}(1))$ defines a general point $x \in \mathrm{Pic}_{14,8}$. One has $h^0(\mathcal{I}_{B'}(2)) = 7$. Then, on an open set of $\mathrm{Pic}_{14,8}$, there exists a Grassmann bundle with fibre $G(5, 7)$

$$\phi : \mathcal{V}_{14} \to \mathrm{Pic}_{14,8}$$

such that $\phi^{-1}(x) = G(5, H^0(\mathcal{I}'_B(2))$. Notice that, counting dimensions,

$$\dim\ \mathcal{V}_{14} = \dim\ \mathcal{W}^1_{8,14} = \dim\ \mathcal{M}_{14}.$$

The linkage now defines a rational map

$$\psi : \mathcal{W}^1_{8,14} \dashrightarrow \mathcal{V}_{14}.$$

Indeed $\Lambda := H^0(\mathcal{I}_C(2))$ is 5-dimensional and the base locus of $|\mathcal{I}_C(2)|$ is $B \cup C$. In particular Λ defines a point of the Grassmannian $G(5, H^0(\mathcal{I}_C(2)))$. Hence $(B, \mathcal{O}_B(1), \Lambda)$ defines a point of $\mathcal{V}_{14}$. The construction is clearly modular. By definition, ψ is the induced map of moduli spaces. Note that ψ is invertible onto its image. Indeed the triple $(B, \mathcal{O}_B(1), \Lambda)$ uniquely defines C and $\omega_C(-1) = L$. Since $\mathcal{W}^1_{8,14}$ and $\mathcal{V}_{14}$ are irreducible of the same dimension, it follows that ψ is birational. Since $\dim\ G(5, 7) = 10$, we conclude that $\mathcal{W}^1_{8,14} \cong \mathrm{Pic}_{14,8} \times \mathbf{P}^{10}$.

(1.2) *The case of genus* 12.

The proof of this case is completely analogous. Let $B' \subset \mathbf{P}^6$ a general curve of degree 15 and genus 8 such that $(B', \mathcal{O}_{B'}(1))$ defines a general point $x \in \mathrm{Pic}_{14,8}$. One has $h^0(\mathcal{I}_{B'}(2)) = 6$. Then there exists a $\mathbf{P}^5$-bundle

$$\phi : \mathcal{V}_{12} \to \mathrm{Pic}_{15,9}$$

such that $\phi^{-1}(x) = \mathbf{P}H^0(\mathcal{I}_{B'}(2))^*$. Again one defines by linkage a map

$$\psi : \mathcal{W}^0_{5,12} \to \mathcal{V}_{12}.$$

By definition ψ sends the moduli point of (C, L) to the moduli point of $(B, \mathcal{O}_B(1))$, where $B \cup C$ is the base locus of $|\mathcal{I}_C(2)|$. One can show as above that ψ is birational. Hence it follows that $\mathcal{W}^0_{5,12} \cong \mathrm{Pic}_{15,9} \times \mathbf{P}^5$.

(1.2) *The case of genus* 15.

By [BV] we have $h^0(\mathcal{I}_C(2)) = 4$ and $C \subset S \subset \mathbf{P}^6$, where S is a smooth (2,2,2,2) complete intersection. Moreover dim $|C| = 2$ and the image of $|C|$ in $\mathcal{W}^1_{9,15}$, via the natural map, is a rational surface R. We can consider the Noether-Lefschetz family of smooth complete intersections $S \subset \mathbf{P}^6$ of type (2,2,2,2) such that Pic S is generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(D)$, where D is a smooth, integral curve of the Hilbert scheme of C. Let $\mathcal{S}$ be the GIT quotient of such a family. We have a dominant rational map $f : \mathcal{W}^1_{9,15} \dashrightarrow \mathcal{S}$ sending the moduli point of (C, L) to the class of S in $\mathcal{S}$. Since S is the base locus of $|\mathcal{I}_C(2)|$, the fibre of f at the moduli point of $S$is the rational surface R.

(2) <u>*Genus* 13 *and* 11</u>

(2.1) *The case of genus* 13.

Let $\mathcal{N}_{13,32}$ be the irreducible family of 32-nodal plane curves of degree 11 and genus 13. We consider pairs (Γ, o) such that $o \in \mathrm{Sing}\ \Gamma$ and $\Gamma \in \mathcal{N}_{13,32}$.

Let $\nu : C \to \Gamma$ be the normalization map, $L := \nu^*\mathcal{O}_{\mathbf{P}^2}(1)$ and $n := \nu^* o$. Then (C, L) defines a general point of $\mathcal{W}^2_{11,13}$. We denote by $\tilde{\mathcal{W}}^2_{11,13}$ the moduli space of triples (C, L, n) and consider the natural forgetful map

$$f : \tilde{\mathcal{W}}^2_{11,13} \to \mathcal{W}^2_{11,13}.$$

The map f has degree 32. We want to construct in our usual way a birational isomorphism

$$\tilde{\mathcal{W}}^2_{11,13} \dashrightarrow \mathrm{Pic}_{12,8} \times \mathbf{P}^{12}.$$

It is standard to check that $H := \omega_C(-L) \otimes \mathcal{O}_C(n)$ defines a morphism

$$h : C \to C_n \subset \mathbf{P}^4$$

which is generically injective. Since $h^0(H(-n)) = 4$, it follows that the curve $C_n := h(C)$ is 1-nodal and its only node is $h(n)$. It is known that C_n is linked to a projectively normal curve B of degree 12 and genus 8 by a (3,3,3) nodal complete intersection, see [Ve1] section 10.

Since $h^0(\mathcal{I}_B(3)) = 6$, it follows that there exists a *unique* $F_n \in |\mathcal{I}_B(3)|$ having multiplicity two at $o := h(n)$. Note that this property is satisfied by

a general pair $(B, o) \in \mathcal{H} \times \mathbf{P}^4$, where $\mathcal{H}$ is an irreducible open neighborhood of B, in its Hilbert scheme, parametrizing smooth curves.

Notice also that the family of projectively normal curves of genus 8 and degree 12 is irreducible and dominates $\mathrm{Pic}_{12,8}$. Hence we can assume that $(B, \mathcal{O}_B(1))$ defines a general point $x \in \mathrm{Pic}_{12,8}$.

Now let $p \in \mathbf{P}^4$ be general and let $F_p \in |\mathcal{I}_B(3)|$ be the unique cubic with multiplicity two at p. Let $V_p \subset H^0(\mathcal{I}_{B/F_p}(3))$ be the subspace of sections vanishing at p. Then, on an open set of $\mathbf{P}^4$, there exists a natural Grassmann bundle G_B, whose fibre at the point p is the Grassmannian $G(2, V_p)$. Hence the construction defines a dominant rational map

$$\phi : \mathcal{V}_{13} \dashrightarrow \mathrm{Pic}_{12,8},$$

with fibre G_B at the moduli point x. The proof then goes as previously: let (C, L, n) be as above. Keeping our notations, we have a unique cubic F_o, containing C_n and which is singular at o. Let $\Lambda \subset H^0(\mathcal{I}_{F_o/B}(3))$ be the 2-dimensional image of $H^0(\mathcal{I}_{C_n}(3))$ via the restriction map. Then Λ is an element of G_B. Hence the assignement $(C, L) \mapsto (B, \mathcal{O}_B(1), \Lambda)$ defines a rational map

$$\psi : \tilde{\mathcal{W}}^2_{11,13} \dashrightarrow \mathcal{V}_{13}.$$

It turns out that (C, L) is uniquely reconstructed from the triple $(B, \mathcal{O}_B(1), \Lambda)$. Hence ψ is generically injective. Since it is a map between varieties of the same dimension, then ψ is birational. This implies that

$$\tilde{\mathcal{W}}^2_{11,13} \cong \mathrm{Pic}_{12,8} \times \mathbf{P}^4 \times G(2,4) \cong \mathrm{Pic}_{12,8} \times \mathbf{P}^8.$$

Remark 2.3. We will see that $\mathrm{Pic}_{12,8}$ is unirational. Hence it follows that: *the family $\mathcal{N}_{13,32}$ of nodal plane curves of genus 13 and degree 11 is unirational.* By proposition 2.11 it is not ruled by linear spaces.

(2.1) *The case of genus* 11.

We simply summarize our usual recipe for this case. Assume (C, L) defines a general point of $\mathcal{W}^0_{6,11}$. Then $\omega_C(-L)$ embeds C in $\mathbf{P}^4$ as a projectively normal curve. Moreover C is linked to a projectively normal curve B, of genus 9 and degree 13, by a (3,3,3) complete intersection. Then, over an open set of $\mathrm{Pic}_{13,9}$, one has a $\mathbf{P}^3$-bundle $\phi : \mathcal{V}_{11} \to \mathrm{Pic}_{13,9}$, with fibre $|\mathcal{I}_B(3)|^*$ at the moduli point of $(B, \mathcal{O}_B(1))$. By linkage we can associate to (C, L) the triple $(B, \mathcal{O}_B(1), P)$, where $P := |\mathcal{I}_{C \cup B}(3)|$ is an element of $|\mathcal{I}_B(3|^*$. This defines a birational map $\psi : \mathcal{W}^0_{6,11} \to \mathcal{V}_{11}$, see [Ve3] section 8. $\square$

2.5. **Unirationality results and rationality issues for $\mathcal{M}_g$.** Finally we review the known rational parametrizations of $\mathcal{M}_g$ and their recent history. After Severi's unirationality results the first new step is due to Sernesi, [Ser1]. In 1981 he proves the unirationality of $\mathcal{M}_{12}$. In 1984 Chang and Ran prove the unirationality of $\mathcal{M}_g$, $g = 11, 12, 13$, [CR1].

The focus was on curves in $\mathbf{P}^3$ and on vector bundles of rank two or higher on $\mathbf{P}^n$, a central topic for that period. Let $C \subset \mathbf{P}^3$ be a smooth,

connected curve of degree d and genus g. The general idea is to represent C as the degeneracy scheme of $(\mathrm{rank} F - 1)$ global sections of a vector bundle F on $\mathbf{P}^3$. Assume d is minimal to have Brill-Noether number $\rho(d, g, 3) \geq 0$ and that C has general moduli. Furthermore let $h^1(\mathcal{I}_C(1)) = h^0(\mathcal{I}_C(2)) = h^0(\omega_C(-2)) = 0$. Then it is shown in [CR1] that a vector bundle F as above always exists as the cohomology sheaf of a complex Γ

$$\mathcal{O}^a_{\mathbf{P}^3}(-1) \to \mathcal{O}^b_{\mathbf{P}^3} \to \mathcal{O}^c_{\mathbf{P}^3}(1),$$

where $a = \rho(d, g, 3)$, $b = 5d - 3g - 17$, $c = 2d - g - 9$, see [CR1] I, Prop. 3 and Remark 3.1. Let $\mathcal{H}_{d,g}$ be an irreducible open set of the Hilbert scheme of $C \subset \mathbf{P}^3$ which dominates $\mathcal{M}_g$ and contains the parameter point of C. Then there exists a quasi-projective variety $\mathcal{F}_{d,g}$ parametrizing triples (Γ, F, Λ) such that (F, Γ) is a pair as above and $\Lambda \subset H^0(F)$ is a vector space of dimension (rank $F - 1$) whose degeneracy scheme is an element of $\mathcal{H}_{d,g}$. Clearly the assignement $(F, \Gamma, \Lambda) \mapsto C$ induces a dominant rational map

$$f_{d,g} : \mathcal{F}_{d,g} \dashrightarrow \mathcal{M}_g.$$

Notice also that $\mathcal{F}_{d,g}$ is a Grassmann bundle over the family $\mathcal{F}$ parametrizing pairs (Γ, F). Thus the unirationality of $\mathcal{M}_g$ follows if $\mathcal{F}$ is unirational. It turns out that this approach to the unirationality problem works for the cases of genus $g = 11, 12, 13$, see [CR1] Proposition 4. Hence we have:

Theorem 2.32. *$\mathcal{M}_g$ is unirational for $g = 11, 12, 13$.*

Remark 2.4. To apply the previous method a necessary condition on F is that $h^1(F) = 0$. Interestingly, this is related to the existence of a quintic surface containing C, see [CR1]I remark 7. Therefore it is related to the possibility that C moves in a linear system on a surface of general type, a question already considered in the previous sections.

Continuing with the recent history we come to the case of genus 14. The unirationality of $\mathcal{M}_{14}$ is a result appeared later, namely in 2005. The method of proof is relatively simple and also applies to the cases of lower genus $g \leq 14$, see [Ve1]. Let us describe it with some more details. From the previous section we have

- ◦ $\mathcal{W}^1_{8,14} \cong \mathrm{Pic}_{14,8} \times \mathbf{P}^{10}$,
- ◦ $\tilde{\mathcal{W}}^2_{15,13} \cong \mathrm{Pic}_{12,8} \times \mathbf{P}^8$,
- ◦ $\mathcal{W}^0_{5,12} \cong \mathrm{Pic}_{15,9} \times \mathbf{P}^5$
- ◦ $\mathcal{W}^0_{6,11} \cong \mathrm{Pic}_{13,9} \times \mathbf{P}^3$

Therefore $\mathcal{M}_g$ is unirational, for $g = 11, 12, 13, 14$, if the universal Picard varieties considered above are unirational. This is true and it can be viewed as a concrete consequence of Mukai's theory of canonical curves of low genus.

A general canonical curve C is a complete intersection for $3 \leq g \leq 5$. This just means that C is a linear section of a suitable Veronese embedding: of $\mathbf{P}^{g-1}$ if $g = 3, 5$ and of a smooth quadric if $g = 4$.

Is it possible to realize a general canonical curve as a linear section of a fixed variety for a few further values of g? The answer is yes when $g = 7, 8, 9$. Let us provide it:

- *For $g = 7, 8, 9$ a general canonical curve C is linear section of a rational homogenous space $S_g \subset \mathbf{P}^{N_g}$.*

More precisely consider the following homogeneous spaces:

(1) $S_7 :=$ *the orthogonal Grassmannian* $OG(5, 10)$ *in* $\mathbf{P}^{15}$*,*

(2) $S_8 :=$ *the Grassmannian* $G(2, 6)$ *in* $\mathbf{P}^{14}$*,*

(3) $S_7 :=$ *the symplectic Grassmannian* $S(3, 6)$ *in* $\mathbf{P}^{13}$*.*

These are subvarieties in their corresponding Grassmannian and we assume that the latter is embedded by its Plücker map. In particular N_g is the dimension of the linear subspace spanned by S_g. Let C be the canonical model of a smooth integral curve of genus $g = 7, 8, 9$. Then:

Theorem 2.33.
- *C is a linear section of S_7 if and only if $W^1_4(C) = \emptyset$,*
- *C is a linear section of S_8 if and only if $W^2_7(C) = \emptyset$,*
- *C is a linear section of S_9 if and only if $W^1_5(C) = \emptyset$.*

The theorem is due to Mukai. It follows from the study of higher rank Brill-Noether theory for a curve C as above. For such a curve there exists a unique vector bundle E, of rank greater than 1 and determinant ω_C, such that $H^0(E)$ defines an embedding of C in S_g as a linear section, cfr [Mu1], [Mu3] and [Mu4]. More precisely one has the following description of E:

$g = 7$: *E is the unique orthogonal rank 5 vector bundle on C such that $h^0(E) = 10$ and*
$detE \cong \omega_C$,

$g = 8$: *E is the unique rank 2 vector bundle on C such that $h^0(E) = 6$ and $\det E \cong \omega_C$,*

$g = 9$: *E is the unique symplectic rank 3 vector bundle on C such that $h^0(E) = 6$.*

We remark that the spaces S_g are rational. Moreover, by the previous description, a set Z of g general points on S_g defines a divisor of degree g on a general curve C of genus g.

Indeed the linear span $\langle Z \rangle$ of Z cuts on S_g a general canonical curve C of genus g and Z turns out to be a general divisor of degree g on C. Using appropriately this remark one obtains:

Theorem 2.34. $\mathrm{Pic}_{d,g}$ *is unirational for $g \leq 9$ and every d.*

Proof. See [Ve1] thm1.2. Let $g = 7, 8, 9$. Consider the open set of elements $z = (z_1, \ldots, z_g) \in S^g_g$ such that $z_1 \ldots z_g$ are linearly independent and the linear span $\mathbf{P}^{g-1}_z$ of them is transversal to S_g. Let

$$\mathcal{C} := \{(x, z) \in S_g \times S^g_g \mid x \in \mathbf{P}^{g-1}_z \cap S_g\}.$$

be the universal canonical section and $p : \mathcal{C} \to U$ its the projection map. Fix $n_1, \dots, n_g$ so that $n_1 + \cdots + n_g = d$ and $n_1 \dots n_g \neq 0$. It is standard to construct a rational map $\phi_d : S_g^g \to \mathrm{Pic}_{d,g}$ sending $z \in U$ is the moduli point of (C, L), where $C = \mathbf{P}_z^{g-1} \cap S_g$ and $L = \mathcal{O}_C(n_1 z_1 + \cdots + n_g z_g)$. On the other hand a general $L \in \mathrm{Pic}^d(C)$ is isomorphic to $\mathcal{O}_C(n_1 z_1 + \cdots + n_g z_g)$ for some $z \in C^g$, cfr. [Ve3] 1.6. Hence ϕ_d is dominant and $\mathrm{Pic}_{d,g}$ is unirational. $\square$

Remark 2.5. The unirationality of $\mathrm{Pic}_{d,g}$ for $g \leq 9$ and each value of d is sharp. Indeed it has been recently proven that the Kodaira dimension of $\mathrm{Pic}_{d,g}$ is not $-\infty$ for each d as soon as $g \geq 10$: see [FV3] and [BFV]. For instance the Kodaira dimension of $\mathrm{Pic}_{g,g}$ is 0 for $g = 10$ and 19 for $g = 11$. Moreover it is $3g - 3$ for $g \geq 12$, [FV3]. If d and $2g - 2$ are coprime, then the same result holds for $\mathrm{Pic}_{d,g}$, [BFV]. As an immediate corollary of the results considered above we have:

Theorem 2.35. *The following Brill-Noether loci are unirational:*

$$\mathcal{W}^1_{8,14},\ \mathcal{W}^2_{11,13},\ \mathcal{W}^0_{5,12},\ \mathcal{W}^0_{6,11}.$$

In particular $\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$.

Remark 2.6. *What about $g = 15, 16$?*

The uniruledness of $\mathcal{M}_{16}$ follows from the general theory on the cone of effective divisors of an algebraic variety: it is proved in [**?**] that the canonical class of $\overline{\mathcal{M}}_{16}$ is not a pseudo-effective class. Then the uniruledness follows from the main result of [BDPP], cfr. [F2] thm. 2.7 .

In [BV] it is proved that $\mathcal{M}_{15}$ is rationally connected. The proof relies again on curves with general moduli moving on surfaces. Indeed a general curve C_i of genus 15 moves in a Lefschetz pencil $P_i \subset |C_i|$ of a smooth (2,2,2,2) complete intersection of $S_i \subset \mathbf{P}^6$, $i = 1, 2$. The image of P_i in $\overline{\mathcal{M}}_{15}$ is a rational curve R_i through the moduli point x_i of C_i. Since P_i is Lefschetz R_i intersects the divisor Δ_0 parametrizing integral nodal curves. It turns out that $y_i \in R_i \cap \Delta_0$ is general in Δ_0. On the other hand Δ_0 is unirational, [BV] thm. 4.3. Then y_1, y_2 are connected by a rational curve R_0 of Δ_0 and $R_0 \cup R_1 \cup R_2$ is a chain of rational curves connecting x_1 to x_2. This implies the rational connectedness of $\overline{\mathcal{M}}_g$.

We want to turn now to a very natural question:

- *What about the rationality of $\mathcal{M}_g$?*

Perhaps it will be considered a surprising phenomenon, in a future history of $\mathcal{M}_g$, that the rationality problem was still unsettled, for many unirational moduli spaces of curves, at the present time. This is however a very difficult problem, often not approachable with the available techniques.

We recall that the rationality of $\mathcal{M}_g$ is known for $g \leq 6$. The case $g = 1$ is classical. Since $\mathcal{M}_1$ is a unirational curve, its rationality also follows from Lüroth theorem. In 1960 J. Igusa proved the rationality of $\mathcal{M}_2$. The

rationality of $\mathcal{M}_4$ and $\mathcal{M}_6$ came later: it was obtained by N. Shepherd-Barron in 1985, [SB]. The rationality of $\mathcal{M}_5$ was then proved by Katsylo [K1] and he also proved the rationality of $\mathcal{M}_3$, cfr. [Bo].

The proof of the rationality of $\mathcal{M}_6$ is related to some arguments considered in this exposition. In particular it deals with 4-nodal plane sextics. Moreover the point of view of the proof has some interplay with the parametrization of the Prym moduli space $\mathcal{R}_6$ considered in later section.

Let $\mathbb{P}$ be the linear system of plane sextics passing with multiplicity ≥ 2 through the points $F_1(1:0:0)$, $F_2(0:1:0)$, $F_3(0:0:1)$, $U(1:1:1)$ of $\mathbf{P}^2$. Consider a general Γ in $\mathbb{P}$ and its normalization $\nu : C \to \Gamma$, then C is a genus 6 curve. Notice also that $\nu^*\mathcal{O}_\Gamma(1) \cong \omega_C(-L)$, where $L \in W^1_4(C)$. We have also seen in section 2 that the degree of the natural map

$$\phi : \mathbb{P} \to \mathcal{M}_6$$

is 120. On the other hand consider the group of Cremona transformations

$$G = \{a \in \mathrm{Bir}(\mathbf{P}^2) \mid \textit{the strict transform of } \mathbb{P} \textit{ by } a \textit{ is } \mathbb{P}\}.$$

G contains the symmetric subgroup S_4 of projective automorphisms fixing the set $\{F_1, F_2, F_3, U\}$. The standard quadratic transformation $q : \mathbf{P}^2 \to \mathbf{P}^2$, centered at F_1, F_2, F_3, is also in G. One can show that

Lemma 2.36. *G is isomorphic to S_5 and it is generated by S_4 and q.*

If $\sigma : S \to \mathbf{P}^2$ is the blow up of F_1, F_2, F_3, U then S is a Del Pezzo surface of degree 5. Moreover it turns out that

$$G = \mathrm{Aut}\ S,$$

the group of biregular automorphisms of S. Then G acts linearly on the strict transform of $\mathbb{P}$ by σ which is $|\omega_S^{-2}|$. In particular it follows that

$$\mathcal{M}_6 \cong |\omega_S^{-2}|/G.$$

The rationality of $\mathcal{M}_6$ then follows because one can prove that the quotient $\mathbf{P}/G$ is rational. This relies on the analysis of the linear representation of G on $H^0(\omega_S^{-2})$, see [SB].

What about the *rationality* of $\mathcal{M}_g$ for $7 \leq g \leq 16$? As already remarked, this appears to be a difficult question. Even the *ruledness* of $\mathcal{M}_g$ seems to be unknown for the same values of g. We have seen examples of ruled universal Brill-Noether loci dominating $\mathcal{M}_g$. We don't know similar examples for $\mathcal{M}_g$ when $7 \leq g \leq 16$. Since rational implies 1-ruled, a criterion of non rationality is the non existence of a ruling of rational curves. So we conclude our discussion with the following problem:

costruct, when it is possible, rulings of $\mathcal{M}_g$ by rational curves.

2.6. **Slope of $\overline{\mathcal{M}}_g$ and related questions.** A large part of the knowledge on the birational geometry of $\mathcal{M}_g$ was made accessible thanks to Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ and to the analysis of its cone of effective divisors. Here a substantial difference appears with respect to classical methods. However, also from this new point of view, the bridge between classical and modern times stays well visible. We end this part of the paper with an account on the slope of the cone of effective divisors of $\overline{\mathcal{M}}_g$, as well as with some free speculations:

Let us recall some basic facts and notations. The boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ is the union of the following irreducible divisors:

(1) Δ_0 whose general point represents an integral, nodal curve of arithmetic genus g with exactly one node,

(2) Δ_i whose general point represents a nodal curve $C_1 \cup C_2$, C_1, C_2 being integral, smooth curves of genus i and $g - i$, where $i = 1 \dots [\frac{g}{2}]$.

By definition δ_i is the class of Δ_i and δ is the class of $\overline{\mathcal{M}}_g - \mathcal{M}_g$. Furthermore we denote by λ the determinant of the Hodge bundle, having fibre $H^0(\omega_C)$ at the moduli point of C. It is well known that $\lambda, \delta_0, \dots, \delta_{[\frac{g}{2}]}$ is a basis of Pic $\overline{\mathcal{M}}_g \otimes \mathbf{R}$. It is a well known fundamental fact that

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta - \delta_1$$

where $K_{\overline{\mathcal{M}}_g}$ is the canonical class of $\overline{\mathcal{M}}_g$ and $g \geq 4$. Let $d = a\lambda - \sum b_i \delta_i \in$ Pic $\overline{\mathcal{M}}_g$ be any divisorial class such that $b_i \geq 0$ for $i = 0, \dots, [\frac{g}{2}]$. Assume that D is a divisor of class d, then we introduce the following:

Definition 2.7. *The slope of D is $s(D) := \frac{a}{b}$ if $b = \min\{b_1 \dots b_{\frac{g}{2}}\} > 0$. For the case $b = 0$ we set $s(D) := \infty$. Moreover the slope of $\overline{\mathcal{M}}_g$ is*

$$s(\overline{\mathcal{M}}_g) \ := \ \min\{s(E), \ / \ E \textit{ is an effective divisor}\}.$$

For the slope of a canonical divisor K of $\overline{\mathcal{M}}_g$ we have of course $s(K) = \frac{13}{2}$. Notice also that $s(E) < \infty$ for every effective E such that $E \cap (\overline{\mathcal{M}}_g - \mathcal{M}_g)$ is non empty: see [HM]. Hence it follows that $mK_{\overline{\mathcal{M}}_g}$ is not an effective class if $s(g) > \frac{13}{2}$ and $m \geq 1$.

Let E be an effective divisor of class $a\lambda - b\delta$ such that $b > 0$. Assume that $R \subset \overline{\mathcal{M}}_g$ is an integral curve moving in a family which covers $\overline{\mathcal{M}}_g$. An obvious remark is that then $ER \geq 0$. But then it follows

$$s(E) = \frac{a}{b} \geq \frac{\delta R}{\lambda R}.$$

Hence lower bounds for the slope of $\overline{\mathcal{M}}_g$ can be obtained by considering curves R moving in a family as above. An effective application of this remark is possible when $R = m(P')$, where P' is a base point free pencil of stable curves of genus g on a smooth surface S'. In such a case one has the well known formulae

$$\lambda R = \chi(\mathcal{O}_{S'}) + g - 1 \ , \ \delta R = c_2(S') + 4(g-1)$$

cfr. [CR2]. It is of course tempting to test them on a K3 surface $(S, \mathcal{O}_S(C))$ of genus g. In this case P' is the strict transform on S' of a pencil $P \subset |C|$ and S' is the blowing up of S at the base locus of P. One computes that

$$\frac{\delta R}{\lambda R} = 6 + \frac{12}{g+1}.$$

This is a fascinating formula because it equals the slope of the canonical class $K_{\overline{\mathcal{M}}_g}$ for $g = 23$. Now $g = 23$ was, before of later results of Farkas, the minimum value of g for which the non uniruledness of $\mathcal{M}_g$ was known. Indeed Farkas proved in [F6] that $\overline{\mathcal{M}}_{23}$ has Kodaira dimension $k(\overline{\mathcal{M}}_{23}) \geq 2$. The slope conjecture, see [HMo], relies on this and further motivations.

Slope conjecture The conjecture says that

$$s(\overline{\mathcal{M}}_g) \leq 6 + \frac{12}{g+1}.$$

The conjecture implies that $\overline{\mathcal{M}}_g$ has Kodaira dimension $-\infty$ for $g \leq 22$. The formula yelds a lower bound to $s(\overline{\mathcal{M}}_g)$ for $g \leq 11$ and $g \neq 10$, because for these values a general C is a hyperplane section of a K3 surface and moves in a pencil P as above. For $g = 10$ the conjecture says that $s(10) = 7 + \frac{1}{11} > 7$. On the other hand the family of stable curves of genus 10 which can be embedded in a K3 surface defines an integral effective divisor

$$E_{K3} \subset \overline{\mathcal{M}}_{10}.$$

This divisor is a first counterexample to the conjecture. In [FP] Farkas and Popa show that $s(E_{K3})$ *is exactly 7*, so that the slope conjecture is false in genus 10. This is a starting point of a series of counterexamples to the conjecture constructed in [F4]. In particular one has:

Theorem 2.37. *The slope conjecture is false for genus $g = 6i + 10$.*

Counterexamples are in this case the divisors $D \subset \overline{\mathcal{M}}_g$ parametrizing the locus of curves C such that, for some $L \in W^1_{3i+6}(C)$, the line bundle $\omega_C(-L)$ does not satisfy the N_i condition of Green-Lazarsfeld, [GL1].

Going back to genus 10 case, we have that E_{K3} is one of these divisors. As pointed out in [FP] the divisor E_{K3} has indeed different incarnations. Let C be a general curve of genus 10 and $L \in W^1_6(C)$. Consider $H := \omega_C(-L)$ and the multiplication map

$$v : \mathrm{Sym}^2 H^0(H) \to H^0(H^{\otimes 2}).$$

Since $h^0(H^{\otimes 2}) = 15$ and $h^0(H) = 5$, it follows that v is a map between vector spaces of the same dimension. Then the condition that v is not an isomorphism is a divisorial condition on $\mathcal{M}_{10}$. On the other hand this is just condition N_1 and characterizes the K3 divisor, see [FP]:

Proposition 2.38. *A stable C as above defines a general point of E_{K3} if and only if v is not an isomorphism for some $L \in W^1_6(C)$.*

Remark 2.7. Let us give, building on the previous properties, a recipe to give a proof that $s(E_{K3}) \leq 7$. We leave to the reader its completion.

Fix a smooth complete intersection of three quadrics $X \subset \mathbf{P}^5$, which contains an integral, non degenerate, sextic elliptic curve E. Then fix a pencil P of degree 2 on E. This defines the rational quartic scroll

$$Y = \cup \langle d \rangle, \ d \in P,$$

where $\langle d \rangle$ is the line spanned by d. Then consider the surface $S = X \cup Y$ and observe that the hyperplane sections of S are reducible, stable curves of arithmetic genus 10 and degree 12. Let $P \subset |\mathcal{O}_S(1)|$ be a general pencil of hyperplane sections and $m : P \to \overline{\mathcal{M}}_{10}$ be the moduli map. Computing as usual the numbers $m^*\delta$ and $m^*\lambda$ one obtains $\frac{m^*\delta}{m^*\lambda} = 7$.

This implies that $s(E) \leq 7$ for every effective divisor E such that $m^{-1}(E)$ is empty. But this is the case when $E = E_{K3}$. Indeed, it is standard to show that the singular surface S is regular and projectively normal. The regularity of S follows from the Mayer-Vietoris type exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_E \to 0,$$

passing to its associated long exact sequence. In a similar way, the property $h^1(\mathcal{I}_S(m)) = 0$, $m \geq 1$, follows from the exact sequence of ideal sheaves

$$0 \to \mathcal{I}_S(m) \to \mathcal{I}_X(m) \oplus \mathcal{I}_Y(m) \to \mathcal{I}_E(m) \to 0.$$

Finally let C be any hyperplane section of S. Then one can deduce that $h^1(\mathcal{I}_C(m)) = 0$, $m \geq 1$, using the standard exact sequence

$$0 \to \mathcal{I}_S(m) \to \mathcal{I}_C(m) \to \mathcal{O}_C(m-1) \to 0.$$

But then the previous map v is an isomorphism for each $C \in P$ and $m^{-1}(E_{K3})$ is empty.

Let us conclude this section by stressing the fact that an important step is still missed in the knowledge of the moduli spaces $\overline{\mathcal{M}}_g$, namely:

What is the Kodaira dimension for $17 \leq g \leq 21$*?*

In recent times, sporadic examples have been discovered of moduli spaces $\mathcal{X}_g$, related to curves of genus g, of intermediate Kodaira dimension. This brings about a change of perspective on moduli spaces related to curves.

Let us denote the Kodaira dimension of $\mathcal{X}_g$ by $k(\mathcal{X}_g)$. We can say that $\mathcal{X}_{g_0}$ *has intermediate Kodaira dimension* if the value of the function $k_{\mathcal{X}}$ is not $-\infty$ nor equal to its maximum.

It would be of course very interesting to discover examples of intermediate Kodaira dimension in the sequence of the moduli spaces $\mathcal{M}_g$.

3. Moduli of spin curves

3.1. **Modern origins and fundamental constructions.** Passing from $\mathcal{M}_g$ to the universal Picard varieties

$$p : \mathrm{Pic}_{d,g} \to \mathcal{M}_g,$$

we want now to consider some remarkable multisections of p. These are the moduli space of pairs (C, L) such that C is a smooth, integral curve of genus g and $L \in Pic^d C$ is a line bundle satisfying some special property. For a multisection as well, we want to discuss the same type of problems considered in the previous section for $\mathcal{M}_g$. The pairs (C, L) to be considered in this section have a name:

Definition 3.1. *A spin curve is a pair (C, L) as above such that L is a theta characteristic.*

Recall that a theta characteristic L is even (odd) if $h^0(L)$ is even (odd). A spin curve (C, L) is said to be *even* (*odd*) if L is even (odd). A modern study of families of spin curves was taken up by Mumford in 1971, [M7]. In particular he proved the following theorem.

Theorem 3.1 (Mumford)**.** *Let $\{(C_t, L_t),\ t \in T\}$ be a family of spin curves. Then $h^0(L_t) \mathrm{mod}\ 2$ is constant on each connected component of T.*

The theorem implies that the moduli space $\mathcal{S}_g$ of spin curves is not connected. It is well known that the connected components of $\mathcal{S}_g$ are exactly two and that they are irreducible. As usual we denote them as

$$\mathcal{S}_g^+ \ ,\ \mathcal{S}_g^- .$$

They are the moduli spaces of even and odd spin curves respectively. The first natural compactifications

$$\overline{\mathcal{S}}_g^+ \ ,\ \overline{\mathcal{S}}_g^-$$

of these moduli spaces were constructed by Cornalba. They are normal projective variety whose fundamental properties are studied in [C]. A further analysis of these spaces and their compactifications can be found in [CC] and [BF]. See also [AJ] for the moduli spaces of generalized spin curves, that is, pairs (C, L) such that L is a p-root of ω_C, for a fixed p.

In order to improve the picture of the birational geometry of $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, it is useful to recall the boundary divisors of Cornalba's compactification, see [C] section 7. One has to consider *semistable* curves C of genus g.

Definition 3.2. *An irreducible component E of a semistable curve C is exceptional if $E = \mathbf{P}^1$ and $E \cap \mathrm{Sing}\ C$ is a set of two points.*

Definition 3.3. *A spin structure on C is a pair (a, L) such that:*
(1) $L \in \mathrm{Pic}\ C$ and $\deg L \otimes \mathcal{O}_E = 1$, for any exceptional component $E \subset C$.
(2) a is a homomorphism $a : L^{\otimes 2} \to \omega_C$ wich is not zero on each irreducible, non exceptional component.

Let $D \subset C$ be the union of the non exceptional components. The locally free sheaf $\mathcal{O}_D(L)$ is a square root of ω_D, see [C] section 2 p.564. L is a theta characteristic if C is smooth. Cornalba's compactification of $\mathcal{S}_g^{\pm}$ is

the moduli space of triples (C, a, L). To have a quick view of the boundary divisors, consider the forgetful map

$$f : \overline{\mathcal{S}}_g^+ \cup \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g,$$

sending the moduli of (C, a, L) to the moduli point of the stable reduction of C. Let $1 \leq i \leq \frac{g}{2}$. For each boundary divisor $\delta_i \subset \overline{\mathcal{M}}_g$ one has:

- $f^*\delta_i \cdot \overline{\mathcal{S}}_g^+ = 2(\alpha_i^+ + \beta_i^+)$,
- $f^*\delta_i \cdot \overline{\mathcal{S}}_g^- = 2(\alpha_i^- + \beta_i^-)$.

where $\alpha_i^+, \beta_i^+, \alpha_i^-, \beta_i^-$ are integral divisors. They are defined as follows. A general x in $f^{-1}(\delta_i)$ is the moduli point of a (C, a, L) such that:

(i) $C = C_1 \cup E \cup C_2$, where C_1, C_2 are smooth, integral curves respectively of genus i and $g - i$, E is an exceptional component;
(ii) if $x \in \alpha_i^+$, ($x \in \beta_i^+$), then L restricts to an even (odd) theta on C_1, C_2;
(iii) if $x \in \alpha_i^-$, ($x \in \beta_i^-$), then L restricts to an even (odd) theta on C_1 and to an odd (even) theta on C_2.

Moreover one has

$$f^*(\delta_0) = (\alpha_0^+ + \alpha_0^-) + 2(\beta_0^+ + \beta_0^-),$$

where $\alpha^{\pm}$, $\beta^{\pm}$ are integral divisors and

$$\alpha_0^+ \cup \beta_0^+ \subset \overline{\mathcal{S}}_g^+ \ , \ \alpha_0^- \cup \beta_0^- \subset \overline{\mathcal{S}}_g^- .$$

Here a general $x \in f^{-1}(\delta_0)$, is a triple (C, a, L) such that:

(i) if $x \in \alpha_0^+ \cup \alpha_0^-$ then C is integral with exactly one node,
(ii) if $x \in \beta_0^+ \cup \beta_0^-$ then $C = D \cup E$, D is integral, E is exceptional.

From now on we denote as

$$f^+ : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g \ , \ f^- : \mathcal{S}_g^- \to \overline{\mathcal{M}}_g,$$

the restrictions of f to $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$. The maps f^+, f^- are finite and their degrees are respectively the numbers of even and odd thetas on a smooth C.

3.2. **The picture of the Kodaira dimension.** The ramification divisor of f is supported on the boundary. Moreover the previous characterizations of the divisors $f^*(\delta_i)$ explicitely describe such a divisor. Therefore, using the formula for the canonical class of $\overline{\mathcal{M}}_g$ and the standard formula for the canonical class of a finite covering, one can compute the canonical classes of $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, cfr. [F1] and [FV3] p.4. Let λ be the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_g$, we fix the further notation:

$$\lambda^{\pm} := f^{\pm *}\lambda.$$

Then the canonical classes we are looking for are the following:

$$K_{\overline{\mathcal{S}}_g^+} \equiv f^{+*}K_{\overline{\mathcal{M}}_g} + \beta_0^+ \equiv 13\lambda^+ - 2\alpha_0^+ - 3\beta_0^+ - 2\sum_{1\le i\le \frac{g}{2}}(\alpha_i^+ + \beta_i^+) - (\alpha_1^+ + \beta_1^+)$$

and

$$K_{\overline{\mathcal{S}}_g^-} \equiv f^{-*}K_{\overline{\mathcal{M}}_g} + \beta_0^- \equiv 13\lambda^- - 2\alpha_0^- - 3\beta_0^- - 2\sum_{1\le i\le \frac{g}{2}}(\alpha_i^- + \beta_i^-) - (\alpha_1^- + \beta_1^-)$$

A systematic approach to the birational geometry of the moduli spaces of spin curves, with special regard to the Kodaira dimension, is due to Farkas. In [F1] some crucial divisorial classes are studied as well as some analogs of the slope conjecture for $\overline{\mathcal{M}}_g$. See also [F3] for a general account on this subject. Among the most interesting divisors of $\overline{\mathcal{S}}_g^+$ one has at least to mention the *spin Brill-Noether divisors* considered in [F5].

Fix $r \geq 0$ such that $(r+1)s = g$ and $d = 2i = r(s+1)$. Then the Brill-Noether number $\rho(r,d,g)$ is zero. Consider the Zariski closure

$$E^{\pm}_{r,d,g} \subset \overline{\mathcal{S}}_g^{\pm}$$

of the moduli of general spin curves (C, L) endowed with some $H \in W^r_d(C)$ satisfying the next condition $\star$. Let $\phi : C \to \mathbf{P}^n$ be the morphism defined by $H \otimes L$, then:

$\star$ *there exists a subspace* Λ *of dimension* $i-2$ *such that* $\phi^*\Lambda$ *has degree* i.

A linear subspace Λ as above is said to be i-secant to the map ϕ. Farkas shows that the $E^{\pm}_{r,g,d}$ is a divisor. The divisors $E^{\pm}_{r,g,d}$ are in some sense analogs of the Brill-Noether divisors of $\overline{\mathcal{M}}_g$. They have good extremality properties in the cone of effective divisors, so one can use them to test the effectivity of the canonical class of $\overline{\mathcal{S}}_g^{\pm}$.

Even spin curves

Building on these methods Farkas proves in [F1] that:

Theorem 3.2. *The moduli space of even spin curves is of general type for* $g \geq 9$ *and uniruled for* $g \leq 7$.

Genus 8

Let us discuss the only left case, namely the case of genus 8. A further very interesting divisor in $\overline{\mathcal{S}}_g^+$ is the theta-null divisor

$$\theta^+_{null}.$$

This is just the Zariski closure in $\overline{\mathcal{S}}_g^+$ of the moduli of spin curves (C, L) such that L is a theta-null, that is, $h^0(L) = 2$. Its image by f is the usual theta-null divisor: $\theta_{null} \subset \overline{\mathcal{M}}_g$. In particular the equivalence

$$\theta^+_{null} \equiv \frac{1}{4}\lambda^+ - \frac{1}{16}\alpha_0^+ - \frac{1}{2}\sum_{i=1\ldots[\frac{g}{2}]}\beta_i \in \operatorname{Pic}\overline{\mathcal{S}}_g^+$$

is a remarkable expression for the class of θ^+_{null}, proved in [F1] theorem 0.2. Note that the formula does not depend on the genus g of C. Using it one computes that in genus 8 the canonical class is effective too. Indeed the formula implies that

$$K_{\overline{\mathcal{S}}^+_8} \equiv c\pi^+ + 8\theta^+_{null} + \sum_{i=1\dots 4} (a_i\alpha^+_i + b_i\beta^+_i),$$

see [F1]. Here c, a_i, b_i are in $\mathbb{Q}^+$ and $\pi^+ := f^*P$, where $P \subset \overline{\mathcal{M}}_8$ denotes the Zariski closure of the moduli of plane curves of degree 7 and geometric genus 8. To conclude the picture, it is shown in [FV3] that:

Theorem 3.3. *The moduli space of even spin curves of genus 8 has Kodaira dimension zero.*

Differently from the case of $\overline{\mathcal{M}}_g$, the description of the Kodaira dimension of $\overline{\mathcal{S}}^+_g$ is therefore complete. Still a natural question is open:

QUESTION: *Does it exists a Calabi-Yau variety birational to $\overline{\mathcal{S}}^+_8$?*

Odd spin curves

Passing to the moduli space $\overline{\mathcal{S}}^-_g$, the complete picture of the Kodaira dimension is obtained in [FV2]. Here there is no intermediate case and the Kodaira dimension assume only two values. Actually one has:

Theorem 3.4. *The moduli space of odd spin curves is of general type for $g \geq 12$ and uniruled for $g \leq 11$.*

The picture of the Kodaira dimension of the moduli of spin curves is therefore complete for every genus. The methods of proof, in the even and the odd case, are in some sense related. Concerning the odd case, it is worth to mention that the proof relies on another effective divisor of $\overline{\mathcal{M}}_g$, different from the divisor θ_{null}. This is the Zariski closure

$$D_g \subset \overline{\mathcal{S}}^-_g$$

of the moduli of pairs (C, L) such that C is smooth and $L \cong \mathcal{O}_C(d)$, where $d = 2x_1 + x_2 + \cdots + x_{g-2}$ is a *singular* effective divisor of C. Computing the class of D_g is an important step, see [FV2] 6.1:

Theorem 3.5. *Let σ be the class of $f^{-*}D_g$ in $\overline{\mathcal{S}}^-_g$. Then:*

$$\sigma = (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{i=1\dots\frac{g}{2}} 2(g-i)\alpha_i - \sum_{i=1\dots\frac{g}{2}} 2i\beta_i.$$

3.3. **K3 surfaces and the uniruledness of $\mathcal{S}^\pm_g$ in low genus.** Once more the uniruledness results for $\overline{\mathcal{S}}^\pm_g$, as well as the Kodaira dimension zero of $\overline{\mathcal{S}}^+_8$, appear as completely related to the world of K3 surfaces.

Let S be a smooth surface and $|H|$ a linear system on S whose general element is a smooth, integral curve. We introduce the following:

Definition 3.4. *A theta pencil of* $(S, |H|)$ *is a triple* (P, Z, E) *such that:*

- $P \subset |H|$ *is a pencil whose general member is smooth, irreducible;*
- Z *is a subscheme of the base locus of* P *and* $E \in \mathrm{Pic}\ S$*;*
- $E \otimes \mathcal{O}_C(Z)$ *is a theta characteristic for any smooth* $C \in P$.

A theta pencil (P, Z, E) *is even (odd) if* $E \otimes \mathcal{O}_C(Z)$ *is even (odd).*

Assume that (C, L) is a spin curve of genus g, defining a *general* point x in the moduli space $\overline{\mathcal{S}}_g$ of even or odd spin curves. The aim of this section is to put in evidence the following, so to say, principle:

$\star$ *x moves in a rational curve of $\overline{\mathcal{S}}_g$ if and only if (C, L) moves in a theta pencil on a K3 surface S.*

Odd theta pencils on K3 surfaces

An easy, but useful, example of theta-pencil is provided by a K3 surface S polarized by $\mathcal{O}_S(H)$. Assume $C \in |H|$ is integral, smooth and let $Z \subset C$ be an effective divisor defining a theta characteristic $L = \mathcal{O}_C(Z)$. Consider the ideal sheaf $\mathcal{I}_Z$ of Z in S and the standard exact sequence

$$0 \to \mathcal{I}_Z(C) \to \mathcal{O}_S(C) \to \mathcal{O}_Z(C) \to 0.$$

From $\deg Z = g - 1$ and $h^0(\mathcal{O}_S(C)) = g + 1$, it follows $h^0(\mathcal{I}_Z(C)) \geq 2$. Let

$$P \subset |\mathcal{I}_Z(C)|$$

be any pencil such that $C \in P$. Then $(P, Z, \mathcal{O}_S)$ is a theta-pencil. More geometrically assume H is very ample and consider the embedding

$$S \subset \mathbf{P}^g,$$

defined by $|H|$. Then C is a hyperplane section of S and a canonical curve. In particular Z spans a linear space Λ of codimension $h^1(\mathcal{O}_C(Z)) + 1 \geq 2$. The pencil P is cut on S by a pencil of hyperplanes through Λ.

In the general case we have $h^0(\mathcal{O}_C(Z)) = 1$, so that $\mathcal{O}_C(Z)$ is an odd theta characteristic. It remains to produce examples of even theta pencils (P, Z, E), such that $E(Z) \otimes \mathcal{O}_C$ is non effective.

Even theta pencils on Nikulin surfaces

Definition 3.5. *A Nikulin surface of genus g is a triple $(S, \mathcal{O}_S(H), E)$, where $(S, \mathcal{O}_S(H))$ is a K3 surface of genus g and $E \in \mathrm{Pic}\ S$ is such that*

- $E^{\otimes 2} \cong \mathcal{O}_S(E_1 + \cdots + E_8)$,
- $E_1, \ldots, E_8$ *are two by two disjoint copies of* $\mathbf{P}^1$,
- $HE = 0$.

Nikulin surfaces are well known, see [N] and [vGS] for a recent account. The divisor $E_1 + \cdots + E_8$ is the branch locus of the finite double covering

$$\hat{\pi} : \hat{S} \to S$$

defined by E. As is well known $\pi^{-1}(E_1), \dots, \pi^{-1}(E_8)$ are exceptional lines and their contraction is a minimal K3 surface $\tilde{S}$ endowed with an involution

$$\iota : \tilde{S} \to \tilde{S}$$

with 8 fixed points. An involution i on a K3 surface X with exactly 8 fixed points is known as a *Nikulin involution*.

Lemma 3.6. *Let $(S, \mathcal{O}_S(H), E)$ be a Nikulin surface of genus $g \geq 2$. Then $E \otimes \mathcal{O}_C$ is a non trivial 2-torsion element of* Pic C *for any $C \in |H|$.*

Proof. It suffices to show that $\pi^{-1}(C)$ is connected for any $C \in |H|$. If not we would have $\pi^* C = C_1 + C_2$, where C_1 and C_2 are disconnected copies of C. Since $g \geq 2$, it would follow $C_1^2 = C_2^2 = 2g - 2 > 0$ and $C_1 C_2 = 0$. This contradicts Hodge Index Theorem because then $C_1^2 C_2^2 - (C_1 C_2)^2 > 0$. □

Let $(S, \mathcal{O}_S(H), E)$ be a Nikulin surface of genus $g \geq 2$. To construct some examples of theta pencils such that $E \otimes \mathcal{O}(Z)_C$ is even, fix a smooth $C \in |H|$ and consider $\eta := E \otimes \mathcal{O}_C$.

Via tensor product η defines a permutation p of the set of all thetas of C. Since η is not trivial p is not the identity. Then it is well known that there exists effective divisors $Z \subset C$ such that $\eta(Z)$ is an even theta characteristic. Fixing such a Z we can consider any pencil

$$P \subset |\mathcal{I}_Z(H)|.$$

Then (P, Z, E) is a theta-pencil such that $E \otimes \mathcal{O}_C$ is even. Indeed, by Mumford's theorem stated in 3.1, $E \otimes \mathcal{O}_C(Z)$ is even for each $C \in P$. Notice also that $h^0(E \otimes \mathcal{O}_C)$ is constant too, since it is equal to $h^0(\mathcal{I}_Z(C)) - 1$.

Let (P, Z, E) be a theta pencil on a smooth surface S birational to a K3 surface. Then we have the natural map in the moduli space

$$m : P \to \overline{\mathcal{S}}_g^{\pm}.$$

Definition 3.6. *A K3 rational curve R of $\overline{\mathcal{S}}_g^{\pm}$ is a curve*

$$R = m(P).$$

The above examples are all what we essentially need to prove that:

Theorem 3.7.

(1) *$\mathcal{S}_g^-$ is covered by the family of K3 rational curves for $g \leq 11$,*
(2) *$\mathcal{S}_g^+$ is covered by the family of K3 rational curves for $g \leq 7$.*

In particular:

(1) *$\mathcal{S}_g^-$ is uniruled for $g \leq 11$,*
(2) *$\mathcal{S}_g^+$ is uniruled for $g \leq 7$.*

Proof. To give the proof of the theorem we distinguish the two cases:

(1) *Odd spin curves*

Let (C, L) be a smooth, general odd spin curve of genus g. Then L is isomorphic to $\mathcal{O}_C(Z)$, where Z is effective, supported on $g - 1$ distinct

points and isolated. Assume $g \leq 11, g \neq 10$. From Mukai's description of canonical curves of low genus, we know that there exists an embedding of C in a K3 surface S of genus g so that $\mathcal{O}_S(C)$ is very ample, see [Mu1]. Let $P = |\mathcal{I}_Z(C)|$ be as above. Then $(P, Z, \mathcal{O}_S)$ is a theta pencil and $m(P)$ is a K3 rational curve through the moduli point of (C, L).

Since a general smooth curve C of genus 10 does not embed in a K3 surface, the latter argument does not work for $g = 10$. However, let (C, x, y) be a general 2-pointed curve of genus 10. It follows from [FKPS] theorem 5.1 and remark 5.2 that:

Proposition 3.8.

(1) *C embeds in a surface S which is a K3 blown up in one point,*
(2) *$x + y = C \cdot E$, where E is the exceptional line in S,*
(3) *the image of C in the minimal model of S is very ample.*

Now assume that $x, y \in Z$, where $L := \mathcal{O}_C(Z)$ is an odd theta. From $\mathcal{O}_C(C + E) \cong \omega_C$ and $\mathcal{O}_C(E) \cong \mathcal{O}_C(x + y)$, we have the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \omega_C(-x - y) \to 0.$$

Since S is regular, it follows that there exists a pencil $P \subset |C|$ with base locus the effective divisor $Z - x - y$. In particular $(P, Z - x - y, \mathcal{O}_S(E))$ is a theta-pencil. So it defines a K3 rational curve $m(P)$ passing through the moduli point of (C, L). Assume that (C, L) is a general, odd spin curve of genus 10 and that $x, y \in Z$, where Z is an effective divisor and $L \cong \mathcal{O}_C(Z)$. It follows from [FV2], theorem 3.10 and its proof, that then C is embedded, exactly as in the previous proposition, in a K3 surface S blown in one point. This extends statement (1) to genus 10 and completes the proof for the moduli of odd spin curves.

(2) Even spin curves

In this case we take profit of Nikulin surfaces and their theta pencils. As is well known, the moduli space of Nikulin surfaces of genus g has dimension 11 while $\dim \mathcal{F}_g = 19$. Hence, counting dimensions, we have

Lemma 3.9. *Let C be a general curve of genus g. Then:*

(1) *C does not embed in a K3 surface of genus g if $g \geq 12$,*
(2) *C does not embed in a Nikulin surface of genus g if $g \geq 8$.*

To complete the proof of theorem 3.7, we previously answer the following question. Let C be a smooth, general curve of genus g

- *When there is a Nikulin surface $(S, \mathcal{O}_S(H), E)$ such that $C \in |H|$?*

This is the analog for Nikulin surfaces of the same question for general K3 surfaces of genus g. As we know the answer to the latter one is $g \leq 11$ with the exception $g = 10$. Interestingly, an exception appears in the case of Nikulin surfaces too.

Theorem 3.10. *Let C be a smooth, general curve of genus g. Then there exists a Nikulin surface $(S, |H|, E)$ such that $C \in |H|$ if and only if $g \leq 7$ and $g \neq 6$.*

Proof. We refer to [FV3] for the complete argument. Because of the nice geometry behind it, we sketch the proof in genus 7. Fix a non trivial line bundle L on C such that $L^{\otimes 2} \cong \mathcal{O}_C$. Since C is general $\omega_C \otimes L$ defines an embedding $C \subset \mathbf{P}^5$ of C as a projectively normal curve. In particular one has $h^0(\mathcal{I}_C(2)) = 3$. One can show that

$$C \subset S,$$

where S is a smooth complete intersection of 3 quadrics. Actually, by [FV3] 2.3, S is a Nikulin surface polarized by $\mathcal{O}_S(C)$. To see this consider $E := C - D$, where D is a hyperplane section of S. Then observe that $2E$ is an effective class. This follows considering the standard exact sequence

$$0 \to \mathcal{O}_S(C - 2D) \to \mathcal{O}_S(2C - 2D) \to L^{\otimes 2} \to 0.$$

Since C is quadratically normal, it follows that $h^1(\mathcal{O}_S(2D - C)) = 0$. Then, by Serre duality, we have $h^1(\mathcal{O}_S(C - 2D)) = 0$. Passing to the associated long exact sequence, it follows that $h^0(\mathcal{O}_S(2E)) = h^0(L^{\otimes 2}) = 1$. Let F be the unique element of $|2E|$. Note that $F^2 = -16$ and that $DF = 8$. A more careful analysis shows that $F = F_1 + \cdots + F_8$, the summands being two by two disjoint lines. Hence $(S, \mathcal{O}_S(C), E)$ is a Nikulin surface of genus 7. □

Remark 3.1. Consider a general Nikulin surface $(S, \mathcal{O}_S(C), E)$ of genus 6. It is possible to show that $|C - E|$ defines an embedding $S \subset \mathbf{P}^4$ so that S is a complete intersection of type (2,3). Then it follows $\mathcal{O}_C(1) \cong \omega_C(E)$. Moreover $L := \mathcal{O}_C(E)$ is a non trivial element of $\mathrm{Pic}^0_2(C)$. In other words C is embedded in $\mathbf{P}^4$ as a Prym canonical curve, (see the next section 4).

It is well known that a general Prym canonical curve C of genus 6 is quadratically normal. This follows, for instance, from the description of the Prym map in genus 6 and of its ramification, see [DS] section 4. In particular this implies that no quadric contains C and contradicts the existence of S. This also explains why the theorem fails in genus 6.

We can now complete the proof of theorem 3.7 for even spin curves:

Let (C, L) be a general even spin curve of genus $g \leq 7$, $g \neq 6$. Then there exists a Nikulin surface $(S, |H|, E)$ so that $\eta \cong E \otimes \mathcal{O}_C$. Fix $\eta := L(-Z)$, where $Z \subset C$ is an odd theta characteristic. Let $P = |\mathcal{I}_Z(C)|$, then (P, Z, E) is a theta pencil on S, as in the example, and $m(P)$ passes through the moduli point of (C, L). It remains to show that $\mathcal{S}_g^+$ is covered by K3 rational curves when $g = 6$. Replacing the word 'K3 surface' by 'Nikulin surface', the proof is completely analogous to the one considered when $g = 10$ in the proof of proposition 3.8. We omit further details about this case. □

Finally we remark that the $\star$ principle, mentioned at the beginning of this section, holds true: $\mathcal{S}_g^{\pm}$ is uniruled if and only if it is covered by K3 rational curves. This is a difference with respect to $\mathcal{M}_g$ as soon as $g \geq 12$.

3.4. **Geometry of the moduli of spin curves in genus 8.** We have already seen that the geometry of curves of genus 8 is in many ways related to the study of other moduli spaces of curves of low genus. Such an experimental fact is confirmed for the moduli of spin curves. So there are good reasons to concentrate on the genus 8 geometry.

In this section we prove the unirationality of $\mathcal{S}_8^-$, which is representative of other unirationality results for $\mathcal{S}_g^-$. Then we will discuss the transition from negative to non negative Kodaira dimension in the even and odd cases.

Canonical curves of genus 8

A crucial property is the realization of a general canonical curve C of genus 8 as a linear section of the Plücker embedding $G \subset \mathbf{P}^{14}$ of the Grassmannian of lines of $\mathbf{P}^5$. We recall something more on this, see [Mu1] and [Mu4].

Let C be general of genus 8, so that $W^1_5(C)$ is finite. Then there exists a unique rank 2 vector bundle E on C such that $\det E \cong \omega_C$ and $h^0(E) = 6$. Such an E fits in an exact sequence

$$0 \to A \to E \to \omega_C(-A) \to 0,$$

where $A \in W^1_5(C)$. This is uniquely defined by some $e \in \mathbf{P}Ext^1(\omega_C(-A), A)$. As we will see, it turns out that E does not depend on the choice of A in $W^1_5(C)$. Let $V = H^0(E)^*$. Then E defines a map

$$f_E : C \to G \subset \mathbf{P}(\wedge^2 V),$$

where G is the Plücker embedding of the Grassmannian $G(2, V)$. $f_E(x)$ is the point represented by the vector $\wedge^2 E^*_x$. One can show the surjectivity of the natural determinant map

$$d : \wedge^2 H^0(E) \to H^0(\omega_C).$$

Let $K^\perp \subset \wedge^2 V$ be the space orthogonal to $K := \mathrm{Ker}\ d$, then $\mathbf{P}K^\perp$ is the linear span of $f_E(C)$. Since d is surjective, $f_E : C \to \mathbf{P}K^\perp$ is the canonical embedding. Let us identify C to $f_E(C)$. Then we conclude that

$$C \subseteq \mathbf{P}K^\perp \cap G.$$

Theorem 3.11 (Mukai)**.** *Let C be a smooth curve of genus g. The following condition are equivalent*

(1) *$W^2_7(C)$ is empty,*
(2) *$C = \mathbf{P}K^\perp \cdot G$.*

Remark 3.2. Let $G^* \subset \mathbf{P}(\wedge^2 V)^*$ be the dual Grassmannian. One expects that $\mathbf{P}K \cdot G^*$ is 0-dimensional of length 14. This is indeed the degree of G^* and it is also the length of $W^1_5(C)$. Actually there is a biregular map

$$w : \mathbf{P}K \cdot G^* \to W^1_5(C).$$

defined as follows: let $b_1 \wedge b_2$ be a decomposable, non zero vector defining a point $b \in \mathbf{P}K$. Then b_1, b_2 generate a line budle $B \subset E$. One can compute that $B \in W^1_5(C)$. Then, by definition, $w(b) = B$. The inclusion $B \subset E$ induces an exact sequence

$$0 \to B \to E \to \omega_C(-B) \to 0.$$

In particular E is independent on the choice of B in $W^1_5(C)$, cfr. [Mu4].

Let us fix from now on the following notations:

Definition 3.7. *$\overline{\mathcal{C}}$ is the family of linearly normal, stable canonical curves of genus 8 in G. $\mathcal{C}$ denotes the open subset parametrizing smooth curves.*

We want to show that

Theorem 3.12. *$\mathcal{S}^-_8$ is unirational.*

We outline the proof given in [FV2]. To this purpose we consider the moduli map $m : \overline{\mathcal{C}} \to \overline{\mathcal{M}}_8$. Due to the mentioned results of Mukai, it is known that m factors through the quotient map $\overline{\mathcal{C}} \to \overline{\mathcal{C}}/Aut\ G$ and a dominant birational morphism $\overline{m} : \overline{\mathcal{C}}/Aut\ G \to \overline{\mathcal{M}}_8$. Let us introduce some preliminary constructions.

Moduli of 7-nodal elliptic curves
At first we consider the Zariski closure

$$\mathbb{B} \subset \overline{\mathcal{M}}_8$$

of the moduli of 7-nodal integral elliptic curves N. A general N admits an embedding $N \subset G$ as a linear section of G and hence as an element of $\overline{\mathcal{C}}$. This just follows because a general K3 surface S of genus 8 is a linear section of G. Hence $|\mathcal{O}_S(1)|$ contains a 1-dimensional family of integral, stable curves with moduli in $\mathbb{B}$. Notice also that, for a general element N in this family, the seven points of Sing N are linearly independent, cfr. [Ch]. $\mathbb{B}$ is an integral projective variety of dimension 14.

Definition 3.8. *$\mathcal{N} = m^{-1}(U)$, where U is the open set of $\mathbb{B}$ parametrizing curves N such that:*

- *N is an integral linear section of G,*
- *the 7 points of* Sing N *are linearly independent.*

Now we relate the odd spin moduli space $\overline{\mathcal{S}}^-_8$ to a $\mathbf{P}^7$-bundle over U. Let (C, L) be a general odd spin curve of genus 8. We can assume $h^0(L) = 1$ and $L = \mathcal{O}_C(Z)$, where Z is a smooth effective divisor of degree $g-1$. We can also assume that C is canonically embedded as a linear section of G. Consider the universal singular locus

$$\mathcal{S} = \{(N, o) \in \mathcal{N} \times G \ / \ o \in \text{Sing } N\}.$$

and its ideal sheaf $\mathcal{I}$ in $\mathcal{N} \times G$. On $\mathcal{N}$ we have the rank 8 vector bundle

$$\mathcal{E} := p_{1*}\mathcal{I} \otimes p_2^*\mathcal{O}_G(1)$$

where p_1, p_2 are the projection maps of $\mathcal{N} \times G$. The fibre of $\mathcal{E}^*$ at N is $H^0(\mathcal{I}_Z(1))^*$. The projectivization of this vector space is naturally isomorphic to the following family of odd spin curves:

$$\mathbf{P}H^0(\mathcal{I}_Z(1))^* = \{(C, \mathcal{O}_C(Z) \ / \ C \in |\mathcal{I}_Z(1)|^*\}.$$

The next properties are proved in [FV2]:

(1) *$\mathcal{E}$ descends to a vector bundle on U,*
(2) *$\mathcal{N}$ is smooth and irreducible,*
(3) *the natural morphism $\overline{m} : \mathcal{N}/Aut\ G \to \mathbb{B}$ is birational.*

Therefore we conclude that

$$\mathcal{S}_8^- \cong \mathbb{B} \times \mathbf{P}^7.$$

Furthermore we can show the following

Theorem 3.13. *$\mathbb{B}$ is unirational.*

Proof. Consider the correspondence

$$I \subset (\mathbf{P}^2)^7 \times (\mathbf{P}^{2*})^7 \times |\mathcal{O}_{\mathbf{P}^2}(3)|$$

parametrizing triples $(\underline{o}, \underline{l}, E)$ such that:
◦ $\underline{o} = (o_1, \dots, o_7) \in (\mathbf{P}^2)^7$ and the points $o_1, \dots, o_7$ are in general position.
◦ $\underline{l} = (l_1, \dots, l_7) \in (\mathbf{P}^2)^7$ and the lines $l_1, \dots, l_7$ are in general position.
◦ o_i belongs to the line l_i, $i = 1 \dots 7$.
◦ E is a smooth cubic passing through o_i and l_i is transversal to E.

The correspondence I is rational: see [FV2] theorem 4.16 for the details of the proof. Furthermore I is endowed with the rational map $\phi : I \to \mathbb{B}$ sending $(\overline{o}, \overline{l}, E)$ to the moduli point of the 7-nodal curve $\overline{E}$, obtained from E by gluing together the two points of $L_i \cap E$ different from o_i, $i = 1 \dots 7$. It is easy to see that ϕ is dominant. Hence $\mathbb{B}$ is unirational. □

The unirationality of $\mathcal{S}_8^-$ then follows from the theorem, because $\mathcal{S}_8^-$ is birational to $\mathbb{B} \times \mathbf{P}^7$.

3.5. From uniruled to general type: the transition for $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$.

Moduli of even spin curves

For the moduli of even spin curves the transition from the uniruledness to the general type case is represented by $\overline{\mathcal{S}}_8^+$, whose Kodaira dimension is zero. We already remarked that $K_{\overline{\mathcal{S}}_8^+}$ is effective and now we want to see that any positive multiple of it is rigid.

To this purpose we introduce further geometric properties of the family of curvilinear sections of the Grassmannian G. We keep the previous notations. Let X be a projective variety in $\mathbf{P}^n$. In what follows we denote by $\mathcal{I}_X$ the ideal sheaf of X in the linear space spanned by X.

Quadrics through G and special divisors in $\overline{\mathcal{M}}_8$

Special curves of genus 8 are characterized by special properties with respect to the quadrics containing G. Let $C \in \mathcal{C}$, consider the restriction

$$r_C : H^0(\mathcal{I}_G(2)) \to H^0(\mathcal{I}_C(2)).$$

It is useful to remark that r_C is a map between spaces of the same dimension. Let $C \in \mathcal{C}$, we consider two natural divisorial conditions on $\mathcal{C}$:

(1) *r_C is not an isomorphism,*
(2) *an element of $|\mathcal{I}_C(2)|$ is a rank 3 quadric.*

Both conditions (1) and (2) define effective divisors in $\mathcal{M}_8$ and hence in $\overline{\mathcal{M}}_8$, passing to their closure. They can be easily described as follows.

(1) C satisfies (1) if and only if C is not a linear section of G. By Mukai's theorem 3.11 this happens precisely when $W^2_7(C)$ is non empty.

(2) This condition is well known for any genus $g \geq 4$: it characterizes curves C whose moduli point is in the divisor θ_{null}.

As above, π will denote the divisor defined in (1) or its class. π^+ will be its pull-back by the forgetful map $f^+ : \overline{\mathcal{S}}^+_8 \to \overline{\mathcal{M}}_8$.

θ^+_{null} has been already defined, for every genus g, as the locus of moduli of spin curves (C, L) such that L is a theta null.

Geometry of the divisor θ^+_{null}

In genus 8 the theta null condition (2) induces, via the quadrics containing G, an interesting covering family of rational curves

$$R \subset \theta^+_{null}.$$

The geometric reason for the existence of R can be explained as follows. Let (C, L) be a general even spin curve of genus 8 such that L is a theta null. Since π and θ_{null} are distinct irreducible divisors, we can assume that

$$C = G \cdot \langle C \rangle.$$

In $\langle C \rangle$ we have a unique quadric q of rank three containing C. As is well known q is characterized by the condition that its ruling of linear subspaces of maximal dimension cuts the pencil $|L|$ on C. Since (C, L) defines a general point of θ^+_{null}, we can assume that the restriction

$$r_C : H^0(\mathcal{I}_G(2)) \to H^0(\mathcal{I}_C(2))$$

is an isomorphism. Hence there exists exactly one $Q \in |\mathcal{I}_G(2)|$ such that $q = Q \cdot \langle G \rangle$. One can show that Q is smooth, provided (C, L) is sufficiently general, [FV3] proposition 6.1. Consider in the Plücker space $\mathbf{P}(\wedge^2 V)$ the linear subspace

$$\mathbf{P}_q = \cap\, Q_x, \; x \in \text{ Sing } q,$$

where Q_x denotes the tangent hyperplane to the quadric Q at x. Note that $\langle C \rangle \subset Q_x$, since $x \in \text{Sing } q$. Then, under the previous assumptions, one can show that:

(1) *Since Q is smooth, it follows* $\dim \mathbf{P}_q = 9$. *Since* $x \in \text{Sing } q$.

(2) *Since each* Q_x *contains* $\langle C\rangle$, $\langle C\rangle$ *is a hyperplane in* $\mathbf{P}_q$.
(3) $\tilde{q} := Q \cdot \mathbf{P}_q$ *is a quadric of rank 4 and* $\mathrm{Sing}\ \tilde{q} = \mathrm{Sing}\ q$.
(4) $S := G \cdot \mathbf{P}_q$ *is a smooth K3 surface contained in* $\tilde{q}$.

See [FV3] section 6. The rulings of $\tilde{q}$ cut on S two elliptic pencils, say $|F_1|$ and $|F_2|$. They restrict to the same ruling of q. This just means that

$$\mathcal{O}_C(F_1) \cong L \cong \mathcal{O}_C(F_2).$$

One has $\mathcal{O}_S(F_1+F_2) \cong \mathcal{O}_S(1)$ and $|F_1|$, $|F_2|$ define a product map

$$\phi : S \to \mathbf{P}^1 \times \mathbf{P}^1$$

of degree $7 = F_1 \cdot F_2$. In particular ϕ/C is the map defined by $|L|$ and C belongs to I, where

$$I := |\phi^* \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1)|.$$

Moreover each smooth $D \in I$ is a canonical curve contained in a rank 3 quadric, namely the quadric $\tilde{q}\ \cap \langle D\rangle$. As in the case of C, it turns out that ϕ/D is defined by a theta null $L_D \cong \mathcal{O}_D(F_1) \cong \mathcal{O}_D(F_2)$.

The outcome of this construction is clear: let $P \subset I$ be a general pencil containing C, then the base locus of P is $Z+Z'$, for some $Z, Z' \in |L|$. Hence the triple $(P, Z, \mathcal{O}_S(F_i))$ is a theta pencil and

$$R = m(P)$$

is a rational curve in θ^+_{null}, passing through the moduli point x of (C, L).

The next theorem summarizes the property of genus 8 curves with a theta null we have described.

Theorem 3.14. *Let* C *be a general smooth integral curve of genus 8. The following conditions are equivalent:*

- *there exists a theta null* L *on* C,
- $C \in |F_1+F_2|$, *where* $|F_1|$, $|F_2|$ *are distinct pencils of elliptic curves on a K3 surface* S *and* $L \cong \mathcal{O}_C(F_1) \cong \mathcal{O}_C(F_2)$.

The Kodaira dimension of $\overline{\mathcal{S}}^+_8$ *is zero*

Let us consider a general rational curve R of the previous family of rational curves covering θ^+_{null}. This means that $R = m(P)$, where $(P, Z; \mathcal{O}_S(F_i))$ is a theta pencil as above. We begin our discussion on the Kodaira dimension of $\overline{\mathcal{S}}^+_8$ with a result which is sharp and crucial.

Proposition 3.15. $R \cdot \theta^+_{null} = -1$.

The result follows counting with appropriate multiplicity the elements of a general $P \subset I$ which are singular. In particular two elements of P are of type $F_1 \cup F_2$, where F_1, F_2 are integral elliptic curves intersecting transversally in 7 points, see [FV3] lemma 6.6. Since R moves in a family of irreducible curves which cover θ^+_{null}, it follows that

Corollary 3.16. $m\theta^+_{null}$ *is rigid for each* $m \geq 0$.

Remark 3.3. The rigidity of θ^+_{null} can be proved for other moduli $\overline{\mathcal{S}}^+_g$ of even spin curves, namely for $g \leq 9$. The proof relies on a different family of rational curves R' covering the theta null divisor of $\overline{\mathcal{S}}^+_g$, $g \leq 9$. One proves that $R' \cdot \theta^+_{null} = -2$, so that the rigidity follows. The condition $R \cdot \theta_{null} = -1$ is of course sharp in genus 8. It is not clear wether θ^+_{null} is rigid for any g.

An analogous proposition holds true for the effective divisor π^+. This parametrizes nodal plane septics of geometric genus 8. Let $P' \subset |\mathcal{O}_{\mathbf{P}^2}(7)|$ be a general pencil of nodal septics of such a genus. We have again the moduli map $m' : R' \to \overline{\mathcal{M}}_8$ and we can consider the curve

$$R' := f^{+*}(m(P')) \subset \pi^+.$$

One computes that:

Proposition 3.17. $R' \cdot \pi^+ < 0.$

R' moves in a family of irreducible curves covering π^+. Hence it follows:

Corollary 3.18. *π^+ is rigid in $\overline{\mathcal{S}}^+_8$.*

Finally we can summarize the proof of the following:

Theorem 3.19. *The Kodaira dimension of $\overline{\mathcal{S}}^+_8$ is zero.*

Proof. See [FV3], proof of theorem 0.1. We know that the canonical class is effective and that

$$K_{\overline{\mathcal{S}}^+_8} \sim t\theta^+_{null} \ + \ p\pi^+ \ + \sum_{i=1\dots4} a_i A^+_i \ + \ b_i B^+_i,$$

where t, p, a_i, b_i are positive rational numbers. A^+_i, B^+_i are the boundary divisors already considered. We know that there exist covering families of irreducible curves $R \subset \theta^+_{null}$ and $R' \subset \pi^+$ such that $R \cdot \theta^+_{null} < 0$, $R' \cdot \pi^+ < 0$. Moreover it is not difficult to show that

$$R \cdot \pi^+ = R \cdot A^+_i = R \cdot B^+_i = 0 \ , \ R' \cdot \theta^+_{null} = R' \cdot A^+_i = R' \cdot B^+_i = 0.$$

This implies that $m\theta^+_{null}$ and $m\pi^+$ are fixed components of the m-canonical linear system for each $m \geq 1$. To see this consider a general $x \in \theta^+_{null}$. Then x belongs to a curve R of the family considered above. Since $R \cdot mK_{\overline{\mathcal{S}}^+_8} < 0$, it follows that x is in the base locus of $|mK_{\overline{\mathcal{S}}^+_8} vert$. Hence $m\theta^+_{null}$ is a fixed component of $|mK_{\overline{\mathcal{S}}^+_8}|$. The same argument works for $m\pi^+$. It follows that the moving part of the m-canonical linear system is contained in $|m\sum(a_iA^+_i + b_iB^+_i)|$, for $m \geq 1$. But it turns out that the latter is 0-dimensional too. Hence $\dim\ |mK_{\overline{\mathcal{S}}^+_8}| = 0$, for $m \geq 1$. □

Moduli of odd spin curves

The transition from uniruledness to general type has no intermediate case for the moduli of odd spin curves of genus g. The last case where $\overline{\mathcal{S}}^-_g$ is uniruled is for $g = 11$. One would like to understand better some geometric

reasons for the change of the Kodaira dimension from genus 11 to genus 12. We conclude this section with a kind of free speculation on this question.

Let X be a an integral projective variety, one could define the *degree of uniruledness of* X as

$$u(X) := \max \{d \in \mathbb{Z} \ / \ \textit{it exists a generically finite map } f : Y \times \mathbf{P}^d \to X\}.$$

Y is assumed to be integral. Of course Y is not uniruled if $u(X) = 0$. Mumford's conjecture on varieties of Kodaira dimension $-\infty$ says that in this case $u(X)$ is strictly positive . Notice also that X is unirational if and only if $u(X) = \dim X$.

Now let $u(g) = u(\overline{\mathcal{S}}_g^-)$. We have seen that the unirationality of $\mathcal{S}_g^-$ is known for $g \leq 8$ so that $u(g) = 3g-3$ in this case. For $g = 9$ the unirationality of $\mathcal{S}_9^-$ seems plausible, though no complete proof is appeared until now. What about $u(10)$ and $u(11)$?

At least in genus 11 the situation could be quite different. Let us see a possible reason. Consider the moduli space $\mathcal{F}_g$ of K3 surfaces $(S, \mathcal{O}_S(C))$ of genus g and then the universal Hilbert scheme of points

$$q : \mathcal{F}_g[g-1] \to \mathcal{F}_g$$

with fibre at the moduli point of $(S, \mathcal{O}_S(C))$ the Hilbert scheme $S[g-1]$ of 0-dimensional subscheme of length $g-1$. Since $C^2 = 2g-2$ we can define a natural involution

$$i : \mathcal{F}_g[g-1] \to \mathcal{F}_g[g-1].$$

Indeed let x be the moduli point of $(S, \mathcal{O}_S(C), Z)$, where $Z \in S[g-1]$. Then the base locus of $|\mathcal{I}_{Z/S}(C)|$ is $Z + \overline{Z}$ where $\overline{Z} \in S[g-1]$. By definition, $i(x)$ os the moduli point of $(S, \mathcal{O}_S(C), \overline{Z})$. Let

$$\mathcal{T}_g \subset \mathcal{F}_g[g-1]$$

be the locus of fixed points of i. For a general pair $(S, \mathcal{O}_S(C))$ the fibre of $q/\mathcal{T}_g$ at the moduli point of $(S, \mathcal{O}_S(C))$ is the family of the 0-dimensional schemes Z such that $(P_Z, Z, \mathcal{O}_S)$ is an theta pencil of genus g, where

$$P_Z := |\mathcal{I}_{Z/S}(C)|.$$

Equivalently this locus is the closure of the family of all $Z \subset S$ such that Z embeds as an odd theta characteristic in a smooth element of $|C|$. On the other hand let us consider the standard projective bundle

$$p : \mathbb{P} \to \mathcal{K}_{11}$$

with fibre $|C|$ at the moduli point of $(S, \mathcal{O}_S(C))$. Then the pull-back $(q/\mathcal{T}_g)^*\mathbb{P}$ contains a natural $\mathbf{P}^1$-bundle $P \subset \mathbf{P}$, with fibre P_Z at the moduli point of $(S, \mathcal{O}_S(C), Z)$. Note that the diagram

$$\begin{array}{ccccc}
\mathcal{T}_g & \longleftarrow & P & \xrightarrow{m^-} & \mathcal{S}_{11}^- \\
\big\downarrow & & \big\downarrow{\scriptstyle f'^-} & & \big\downarrow{\scriptstyle f^-} \\
\mathcal{K}_{11} & \longleftarrow & \mathbb{P} & \xrightarrow{m} & \mathcal{M}_{11}
\end{array}$$

is commutative and that the vertical arrows f_P and f^- are just the obvious forgetful map. In particular they have the same degree, which is the number of odd theta characteristics on curve of genus 11. Since the moduli map m is birational, the same is true for the moduli map m^-. This shows that

Theorem 3.20. *$\overline{\mathcal{S}}^-_{11}$ is ruled and birational to $\mathcal{T}_{11} \times \mathbf{P}^1$.*

$\mathcal{T}_g$ is a very interesting variety whose Kodaira dimension does not seem to be known. Knowing it is non negative in genus 11 could place $\mathcal{S}^-_{11}$, though uniruled, in an intermediate position between uniruledness and general type.

4. Prym moduli spaces

4.1. **Prym pairs.** Continuing our review of remarkable multisections of $\mathrm{Pic}_{d,g}$, we naturally come to the degree zero universal Picard variety

$$p : \mathrm{Pic}_{0,g} \to \mathcal{M}_g.$$

p is endowed with the zero section $C \to (C, \mathcal{O}_C)$ and the multiplication by n map $\mu_n :\ \mathrm{Pic}_{0,g} \to\ \mathrm{Pic}_{0,g}$. We can define in $\mathrm{Pic}_{0,g}$ the following locus:

Definition 4.1. *$\mathcal{R}_{g,n}$ is the moduli space of pairs (C, L) such that C is a smooth, integral curve of genus g and L is a primitive n-root of $\mathcal{O}_C$.*

In particular $\mathcal{R}_{g,n}$ is a component of the inverse image of μ_n. We recall some constructions related to a pair (C, L). Consider the following sets.

(1) $S_1(C)$: the set of primitive n-roots of $\mathcal{O}_C$;
(2) $S_2(C)$: the set of pairs (π, i) such that $\pi : \tilde{C} \to C$ is a cyclic étale cover of degree n and $i : \tilde{C} \to \tilde{C}$ generates the Galois group of π;
(3) $S_3(C)$: the set of pairs $(\tilde{C}, i)$ such that $i : \tilde{C} \to \tilde{C}$ is an automorphism acting freely on a smooth, integral curve $\tilde{C}$ and $\tilde{C}/\langle i \rangle \cong C$.

Proposition 4.1. *There exist natural bijective maps between the sets $S_1(C)$, $S_2(C)$ and $S_3(C)$.*

Proof. Let $L \in S_1(C)$ and $E := \bigoplus_{i=0\dots n} L^{\otimes i}$. Consider the curve

$$\tilde{C} := \{(x; 1, v, v^{\otimes 2}, \dots, v^{\otimes n-1}) \ / \ v \in L_x \ , \ v^{\otimes n} = 1\} \subset \mathbf{P}E.$$

Then $\tilde{C}$ is smooth, connected and the projection $p : \mathbf{P}E \to C$ restricts to an étale cyclic cover $\pi := p/\tilde{C}$. $\tilde{C}$ is endowed with the obvious automorphism i induced by tensor product with v. This defines a map $u_1 : S_1(C) \to S_2(C)$ such that $u_1(L) := (\pi, i)$. A second map $u_2 : S_2(C) \to S_3(C)$ is defined as follows: $u_2(\pi, i) = (\tilde{C}, i)$. Finally a pair $(\tilde{C}, i) \in S_3(C)$ defines an étale cyclic cover $\pi : \tilde{C} \to C = \tilde{C}/\langle i \rangle$. In particular $\pi_* \mathcal{O}_{\tilde{C}}$ splits as a vector bundle E as above and its summand L is uniquely defined by i. This defines a map $u_3 : S_3(C) \to S_1(C)$ such that $u_3(\tilde{C}, i) = L$. It is easy to see that these maps are bijective. □

From now on we will keep the notations

$$\pi : \tilde{C} \to C \quad , \quad i : \tilde{C} \to \tilde{C}$$

for the maps constructed from a pair (C, L) as above. $\mathcal{R}_{g,n}$ is a very natural multisection and it deserves to be studied in its full generality for every n. Actually only the case $n = 2$ has been object of a detailed study since the seventies of the last century. As exceptions see however [CCC] for the compactification of $\mathcal{R}_{g,n}$ and [LO] for the Prym map associated to $\mathcal{R}_{g,n}$. See also [BC] and [BaV] for the rationality problem when $n = 3$ and $g = 3, 4$. From now on we will assume $n = 2$.

Definition 4.2. *A pair (C, L) as above is called a Prym pair.*

For a Prym pair $\pi : \tilde{C} \to C$ is an étale double covering and $i : \tilde{C} \to \tilde{C}$ is a fixed point free involution. For $\mathcal{R}_{g,2}$ we will use the standard notation

$$\mathcal{R}_g.$$

The space $\mathcal{R}_g$ is well known as the *Prym moduli space.* The special attention owed on Prym pairs is related to the notion of Prym variety, a principally polarized abelian variety of dimension $g - 1$ which is associated to (C, L) when $n = 2$. This bring us to the theory of principaly polarized abelian varieties and to the parametrizations of their moduli spaces.

Postponing further comments, we recall the definition of Prym variety. Let (C, L) be a Prym pair consider the Norm map

$$Nm_0 : \ \mathrm{Pic}^0(\tilde{C}) \to \ \mathrm{Pic}^0(C),$$

sending $\mathcal{O}_{\tilde{C}}(d)$ to $\mathcal{O}_C(\pi_* d)$. Nm_0 is a surjective morphism of abelian varieties. It turns out that the principal polarization of $\mathrm{Pic}^0(\tilde{C})$ restricts to twice a principal polarization Ξ on the connected component A of $\mathrm{Ker}\ Nm_0$, see [M8]. The pair (A, Ξ) is therefore a principally polarized abelian variety:

Definition 4.3. *The Prym variety of (C, L) is the pair (A, Ξ).*

What is known on the existence of rational parametrizations of $\mathcal{R}_g$ can be easily summarized as follows:

Theorem 4.2. *$\mathcal{R}_g$ is unirational for $g \leq 7$ and rational for $g \leq 4$.*

We discuss at first the rationality and then the unirationality results. Once more this gives the opportunity to uncover beautiful constructions and see their common geometric links.

4.2. **Rationality of Prym moduli spaces of hyperelliptic curves.** We start with the *hyperelliptic locus in $\mathcal{R}_g$*. This has many irreducible components. The rationality of all of them, possibly but one, can be deduced from Katsylo's theorem on the rationality of moduli of binary forms, [K1].

Let C be hyperelliptic of genus $g \geq 2$ and let $u : C \to \mathbf{P}^1$ be its associatd double cover. Then the ramification divisor of u is supported on the set

$$W = \{w_1, \ \ldots, \ w_{2g+2}\}$$

of the Weierstrass points of C. Let H be the hyperelliptic line bundle of C and E_t be the family of effective divisors of cardinality $2t$, where $1 \leq t \leq [\frac{g+1}{2}]$, which are supported on $2t$ *distinct* Weierstrass points. For $e \in E_t$ the line bundle $H^{\otimes t}(-e)$ is a square root of $\mathcal{O}_C$, that is an element of $\mathrm{Pic}^0_2(C)$. Let

$$\beta_t : E_t \to \mathrm{Pic}^0_2(C) - \{\mathcal{O}_C\}$$

be the map sending $e \in E_t$ to the line bundle $H^{\otimes t}(-e)$ and let

$$B_t := \beta_t(E_t).$$

The next lemma is standard:

Lemma 4.3.

(1) The map β_t is injective for $t \leq [\frac{g}{2}]$.
(2) For $t = \frac{g+1}{2}$ the map β_t is 2:1 over its image.
(3) $\mathrm{Pic}^0_2(C) - \{\mathcal{O}_C\} = \bigcup_{1 \leq t \leq [\frac{g+1}{2}]} B_t$.

Proof. (1) Let $\beta_t(e) = \beta_t(f)$ for some $e, f \in E_t$, then $\mathcal{O}_C(f) \cong \mathcal{O}_C(e)$. Now e, f are isolated, since $deg\ e \leq g$. Hence $e = f$. (2) Let $e = w_1 + \cdots + w_{g+1}$, it is easy to see that $\beta_t^{-1}(\beta_t(e)) = \{e, w - e\}$, where $w = w_1 + \cdots + w_{2g+2}$. (3) This is an easy well known property as well. □

Definition 4.4. *$\mathcal{RH}^t_g$ is the locus in $\mathcal{R}_g$ of the moduli points of pairs (C, L) such that $L \cong H^{\otimes t}(-e)$ for some $e \in E_t$.*

Let $\mathcal{RH}_g \subset \mathcal{R}_g$ be the hyperelliptic locus, parametrizing pairs (C, L) such that C is hyperelliptic. It is clear that

$$\mathcal{HR}_g = \bigcup_{1 \leq t \leq [\frac{g+1}{2}]} \mathcal{RH}^t_g.$$

Theorem 4.4. *$\mathcal{RH}^t_g$ is rational for $t \leq [\frac{g}{2}]$.*

Proof. Let (C, L) be a Prym pair defining a general $x \in \mathcal{RH}^t_g$. Assume $t \leq [\frac{g}{2}]$. Then, by the previous lemma, there exists a unique $e \in E_t$ such that $L \cong H^{\otimes t}(-e)$. Therefore we can uniquely associate to (C, L) the set $W - Supp\ e$. Consider the 2.1 cover $h : D \to \mathbf{P}^1$, branched on $W - Supp\ e$, then D is hyperelliptic of genus $g - t$. Let $\mathcal{H}_p$ be the moduli space of hyperelliptic curves of genus p. Then we have a dominant rational map

$$\phi : \mathcal{HR}^t_g \to \mathcal{H}_{g-t},$$

induced by the assignement $(C, L) \mapsto D$. As is well known $\mathcal{H}_{g-t}$ is birational to the quotient $\mathbf{P}^1(2g - 2t + 2)/PGL(2)$, where $\mathbf{P}^1(n)$ denotes the n-th symmetric product of $\mathbf{P}^1$. This quotient is rational by Katsilo's theorem, [K1]. Moreover let $\mathbb{P} := (\mathbf{P}^1(2g-2-2t) \times \mathbf{P}^1(2t))/PGL(2)$. It is well known that then $\mathbb{P}$ is a $\mathbf{P}^{2t}$-bundle over an open set $U \subset \mathcal{H}_{g-t}$. Hence $\mathbb{P}$ is rational. Let $x \in \mathbb{P}$ be the orbit of $(p, q) \in \mathbf{P}^1(2g+2-2t) \times \mathbf{P}^1(2t)$. x uniquely defines

a pair (C, L) such that $h : C \to \mathbf{P}^1$ is the 2:1 cover branched on $p + q$ and $L \cong H^{\otimes t}(-e)$, where $e := h^*q$. Hence we have a rational map

$$\psi : \mathbb{P} \to \mathcal{RH}^t_g$$

sending x to the moduli point of (C, L). An inverse to ψ exists: let $y \in \mathcal{HR}^t_g$ be the moduli point of (C, L). Keeping our notations, this uniquely defines $e \in E$ such that $L \cong H^{\otimes t}(-e)$. This isomorphism uniquely defines the orbit $x \in \mathbb{P}$ of the pair (p, q), where $p + q$ is the branch divisor of h and $q = h_*e$. Therefore ψ is generically injective. It is also birational because $\mathbb{P}$ and $\mathcal{HR}^t_g$ are integral of the same dimension. Then $\mathcal{HR}^t_g$ is rational for $t \leq [\frac{g}{2}]$. □

In particular it follows:

Corollary 4.5. *$\mathcal{R}_2$ is rational for $g = 2$.*

Proof. Just observe that $\mathcal{R}_2 = \mathcal{HR}^1_2$. □

$\mathcal{R}_2$ and the Segre primal

A beautiful proof of the rationality of $\mathcal{R}_2$ is due to Dolgachev and relies on Segre cubic primal and its associated geometry. We want to introduce this proof and describe the main geometric constructions behind it. See [D1] and [VdG] for more details. The Segre primal is the cubic threefold V defined in $\mathbf{P}^5$ by the equations

$$y_1 + \cdots + y_5 = y_1^3 + \cdots + y_5^3 = 0.$$

Clearly we have

$$V \subset \mathbf{P}^4 := \{y_1 + \cdots + y_5 = 0\}.$$

V is a nodal cubic threefold with the maximal number of nodes, namely 10, and contains 15 planes. It was discovered by Corrado Segre in [Se]. V is very related to the moduli space $\overline{\mathcal{M}}_2^{(2)}$ of genus 2 curves with a level 2 structure. It is indeed the strict dual of the natural embedding

$$\mathbb{M} \subset \mathbf{P}^{4*}$$

of $\overline{\mathcal{M}}_2^{(2)}$ as a Siegel modular threefold of degree 4. Of course there is a lot of beautiful classical geometry behind this relation, see [VdG]:

(1) Sing $\mathbb{M}$ is the connected union of 15 lines, each of multiplicity 2. This curve has 35 nodes which are triple points for $\mathbb{M}$.

(2) A point $o \in \mathbb{M} - \text{Sing } \mathbb{M}$, represents a genus 2 curve C with a level 2 structure. Let $K_o := T_o \cap \mathbb{M}$, T_o being the tangent hyperplane at o. Then

$$K_o = \text{Pic}^0(C)/\langle -1 \rangle.$$

K_o is a Kummer quartic surface.

(3) Let $q : \text{Pic}^0(C) \to K_o$ be the quotient map, then

$$q^{-1}(\text{Sing } \mathbb{M}) = \text{Pic}^0_2(C) - \{\mathcal{O}_C\}.$$

(4) The forgetful map $f : \mathbb{M} \to \overline{\mathcal{M}}_2$ is the quotient map for the action of $S_6 \cong Sp(4, \mathbb{Z}_2)$ on $\mathbb{M}$. It is induced by the permutations of $(y_1, \dots, y_6)$.

The rationality of $\mathcal{R}_2$

Let T be the set of 15 double lines of $\mathbb{M}$. It follows from (3) that each $o \in \mathbb{M} - \mathrm{Sing}\ \mathbb{M}$ is endowed with a bijection $T \longrightarrow \mathrm{Pic}^0_2(C) - \{\mathcal{O}_C\}$, defined by the assignement $l \mapsto q^{-1}(l)$. We can now deduce the rationality of $\mathcal{R}_2$:

Fix $l \in T$ and consider the map

$$\phi : \mathbb{M} \to \mathcal{R}_2$$

sending o to the moduli point of (C, L), where $L = q^{-1}(l)$. Observe that S_6 acts on T and that ϕ is the quotient map with respect to the stabilizer of l. It is known that this is conjugate to the subgroup $S_4 \times S_2$ of S_6. Therefore we conclude that:

$$\mathcal{R}_2 \textit{ is birational to } \mathbb{M}/S_4 \times S_2.$$

The last step is the rationality of $\mathbb{M}/S_4 \times S_2$, proved in [D1].

4.3. **Rationality of $\mathcal{R}_3$.**

Various proofs of the rationality of $\mathcal{R}_3$ appear to be known since the eighties of the last century. The subject was indeed considered by several authors, see [D1] for some historical account. In particular, the first complete paper containing this result is due to Katsylo, [K2]. The rationality of $\mathcal{R}_3$ stems from the vein of classical geometry we considered in the last part of the previous section. We continue in this vein, keeping the same notations.

$\mathcal{R}_3$ and the Segre primal

Consider the Siegel modular quartic threefold $\mathbb{M} = \overline{\mathcal{M}}_2^{(2)}$ and the sheaf

$$\Omega^1_{\mathbb{M}} \oplus \mathcal{O}_{\mathbb{M}}.$$

The fibre of $\mathbf{P}(\Omega^1_{\mathbb{M}} \oplus \mathcal{O}_{\mathbb{M}})$ at any smooth $o \in \mathbb{M}$ is the dual of the tangent hyperplane to $\mathbb{M}$ at o. In other words we have an obvious identification

$$\mathbf{P}(\Omega^1_{\mathbb{M}} \oplus \mathcal{O}_{\mathbb{M}})_o = |\mathcal{O}_{K_o}(1)|,$$

where $K_o = \mathrm{Pic}^0(C)/\langle -1 \rangle$ is the Kummer quartic surface considered in the previous section. Let $f : \mathbb{M} \to \overline{\mathcal{M}}_2$ be the forgetful map. S_6 acts linearly and almost freely on the hypersurface $\mathbb{M}$. Hence it acts as well on the fibres of $\mathbf{P}(\Omega^1_{\mathbb{M}} \oplus \mathcal{O}_{\mathbb{M}})$ and descends to a $\mathbf{P}^3$-bundle $u : \mathbb{K} \to U$ over an open set U of $\mathcal{M}_2$. Let $p = f(o) \in \mathcal{M}_2$, we point out that there exists a well defined surface which is naturally associated to $\mathbb{K}_p$. This is the dual

$$D_p \subset \mathbb{K}_p$$

of K_o. It is well known that D_p is the union of 16 planes, corresponding to the 16 nodes of K_o, and of the strict dual surface

$$\hat{K}_p \subset \mathbb{K}_p$$

of K_o. This is a Kummer quartic surface again. Note that any

$$H \in \mathbb{K}_p - D_p$$

is a smooth plane section of K_o. Moreover H is endowed with a non split étale double covering $\pi : \tilde{H} \to H$. Indeed let C be the moduli point of p, then π is induced by the quotient map $\mathrm{Pic}^0(C) \to K_o$. Let $L_H \in \mathrm{Pic}^0_2(C)$ be the line bundle defining π, we have reconstructed from (H, p) a Prym pair (H, L_H) and hence a point of $\mathcal{R}_3$. This defines a rational map

$$\phi : \mathbb{K} \to \mathcal{R}_3.$$

Its description ϕ follows from [Ve2]: ϕ fits in the commutative diagram

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\phi} & \mathcal{R}_3 \\ \big\downarrow u & & \big\downarrow p_3 \\ \mathcal{M}_2 & \xrightarrow{j} & \mathcal{A}_2, \end{array}$$

where $p_3 : \mathcal{R}_3 \to \mathcal{A}_2$ is the Prym map and $j : \mathcal{M}_2 \to \mathcal{A}_2$ is the Torelli map. It turns out that the fibre of the compactified Prym map at $j(p)$ is a suitable modification of the quotient space

$$\mathbb{K}_p/\mathbb{Z}_2^4.$$

Here the group $\mathbb{Z}_2^4$ acts on $\mathbb{K}_p = \mathbf{P}^3$ as the group of projective automorphisms induced by the group of translations on $\mathrm{Pic}^0(C)$ by 2-torsion elements. It is a well known property, from classical theta functions theory, that

$$\mathbf{P}^3/\mathbb{Z}_2^4 = \mathbb{M}^*.$$

$\mathbb{M}^*$ denotes the Segre cubic primal, that is, the strict dual of $\mathbb{M}$. This is shown in Hudson's book [Hu], where $\mathbb{P}^3/\mathbb{Z}_2^4$ is explicitely described. This is the image of the map $q : \mathbf{P}^3 \to \mathbf{P}^4$ defined by the linear system of the $\mathbb{Z}_2^4$-invariant quartic surfaces

$$a(x_1^2x_4^2+x_2^2x_3^2)+b(x_2^2x_4^2+x_1^2x_3^2)+c(x_1^2x_4^2+x_2^2x_3^2)+2dx_1x_2x_3x_4+e\sum x_i^4 = 0.$$

Its cubic equation is

$$4e^3 - a^2e^-b^2e - c^2e + abc + d^2e = 0.$$

It is shown in [Ve2] that $q = \phi/\mathbb{K}_p$. Then we can conclude that

Theorem 4.6. *$\mathcal{R}_3$ is birational to a fibration over $\mathcal{M}_2$ with fibre $\mathbb{M}^*$.*

$\mathcal{R}_3$ and the Coble-Roth map

A proof of the rationality of $\mathcal{R}_3$, due to Dolgachev, relies on the classical Coble-Roth map. We want to introduce it and then summarize such a proof. Let

$$\mathcal{R}_4^b$$

be the moduli space of Prym pairs (F, L_F) such that F is a bielliptic curve of genus 4 and $L_F \in \sigma^* \operatorname{Pic}^0_2(E)$, where $\sigma : F \to E$ is a bielliptic map. For proving the rationality of $\mathcal{R}_3$, the main tool used is a rational map

$$\phi : \mathcal{R}_3 \to \mathcal{R}^b_4.$$

This map is classical, it is defined in [D1] as the *Coble-Roth map.* The rationality of $\mathcal{R}_3$ is then a consequence of the following steps:

Theorem 4.7.

(1) *ϕ is birational,*
(2) *$\mathcal{R}^b_4$ is rational.*

This theorem is due to Dolgachev, [D1]. We describe here the proof of (1). This is an almost immediate consequence of the existence of ϕ.

Let (C, L) be a *general* Prym pair, defining a point of $\mathcal{R}_3$. Since $L^{\otimes 2}$ is trivial, the assignement $N \mapsto L \otimes N$ defines a fixed points free involution

$$t_L : \operatorname{Pic}^2(C) \to \operatorname{Pic}^2(C).$$

In $\operatorname{Pic}^2(C)$ consider the theta divisor $W^0_2(C) = C^{(2)}$ and the curve

$$\tilde{F} := \{x + y \in C^{(2)} \ / \ h^0(\omega_C \otimes L(-x-y) \geq 1\}.$$

$\tilde{F}$ is obtained from the 4-gonal pencil $|\omega_C \otimes L|$ via Recillas' construction, [Re]. It turns out that $\tilde{F}$ is a smooth, integral curve of genus 7. Moreover it is easy to check that $\tilde{F} = W^0_2 \cap t_L(W^0_2)$. Hence t_L induces a fixed points free involution $i_{\tilde{F}}$ on $\tilde{F}$. Then its quotient map is an étale double covering

$$\pi_F : \tilde{F} \to F,$$

induced by some L_F in $\operatorname{Pic}^0_2(F)$. Recillas' construction also implies that:

Proposition 4.8. *The Prym variety of (F, L_F) is $\operatorname{Pic}^0(C)$.*

On $\tilde{F}$ we have a second involution $j : \tilde{F} \to \tilde{F}$. By definition this associates to $x + y \in \tilde{F}$ the unique $z + t$ such that $x + y + z + t$ is a canonical divisor. Hence $x + y$ is fixed by j if and only if $\mathcal{O}_C(x + y)$ and $L(x + y)$ are odd theta characteristics. The formula for the number of these odd thetas gives 12. Let

$$\tilde{\sigma} : \tilde{F} \to \tilde{E} := \tilde{F}/\langle j \rangle$$

be the quotient map. Then, by Hurwitz formula, $\tilde{E}$ is elliptic. It is easy to see that $i_{\tilde{F}}$ and j commute. Moreover $i_{\tilde{F}}$ induces a fixed points free involution $i_{\tilde{E}}$ on $\tilde{E}$, so that $E = \tilde{E}/\langle i_{\tilde{E}} \rangle$ is elliptic. Hence we have the commutative diagram

$$\begin{tikzcd} \tilde{F} \arrow[r, "\pi_F"] \arrow[d, "\tilde{\sigma}"] & F \arrow[d, "\sigma"] \\ \tilde{E} \arrow[r, "\pi_E"] & E \end{tikzcd}$$

where the vertical arrows are the quotient maps of $i_{\tilde{F}}$ and $i_{\tilde{E}}$ respectively. It follows from the diagram that $L_F \in \sigma^* \operatorname{Pic}^0_2(E)$. Hence the Prym pair (F, L_F) defines a point of the Prym moduli space of bielliptic curves $\mathcal{R}^b_4$.

Definition 4.5. *The Coble-Roth map $\phi : \mathcal{R}_3 \to \mathcal{R}^b_4$ is the map sending the moduli point of (C, L) to the moduli point of (F, L_F).*

Using the diagram, we can now define a rational map

$$\psi : \mathcal{R}^b_4 \to \mathcal{R}_3$$

inverse to ϕ. Starting from (F, L_F) we can reconstruct at first the previous commutative diagram. Moreover $\operatorname{Pic}^0(C)$ is the Prym of (F, L_F) and sits in $\operatorname{Pic}^0(\tilde{F})$ as the connected component of zero of the Kernel of the Norm map $\pi_{F*} : \operatorname{Pic}^0(\tilde{F}) \to \operatorname{Pic}^0(F)$. On the other hand

$$\tilde{\sigma}^* : \operatorname{Pic}^0(\tilde{E}) \to \operatorname{Pic}^0(\tilde{F}) \text{ and } \sigma^* : \operatorname{Pic}^0(E) \to \operatorname{Pic}^0(F)$$

are injective because $\tilde{\sigma}$ and σ are ramified, cfr. [M8]. It follows from the previous commutative diagram, since $\tilde{\sigma}_* \tilde{\sigma}^*$ is just multiplication by 2, that $\tilde{\sigma}^* \operatorname{Pic}^0_2(\tilde{E})$ is contained in Ker π_{F*} and finally that

$$\operatorname{Pic}^0_2(C) \cap \tilde{\sigma}^* \operatorname{Pic}^0_2(\tilde{E}) = \pi_F{}^* \operatorname{Pic}^0_2(E) \cong \mathbb{Z}_2.$$

Furthermore, it turns out that $\pi^*_F \operatorname{Pic}^0_2(E)$ is generated by L. Starting from (F, L_F), we have reconstructed (C, L). It follows that the assignement $(F, L_F) \mapsto (C, L)$ defines a map ψ which is inverse to the Coble-Roth map ϕ. This implies that ϕ is birational, see [D1] for further details on this proof and for the proof that $\mathcal{R}^b_4$ is rational.

4.4. **Rationality of $\mathcal{R}_4$.** As we already have seen, Prym moduli spaces in very low genus are frequently related to linear systems $|C|$ of smooth, connected curves on a surface S endowed with a quasi étale double cover

$$p : \tilde{S} \to S.$$

An example in genus 3 is the Kummer quartic surface $S \subset \mathbf{P}^3$. In this case p is the 2:1 cover of S branched on its 16 nodes of S and $|C| = |\mathcal{O}_S(1)|$. We are going to meet further examples when considering $\mathcal{R}_g$ for $g = 4, 5, 6$.

The rationality of $\mathcal{R}_4$ is due to Catanese, [Ca]. In this case S is a fixed surface in $\mathbf{P}^3$, up to projective equivalence. Namely S is the 4-nodal cubic surface, known also as *Cayley cubic.* Fixing on $\mathbf{P}^3$ suitable coordinates $(x_1 : x_2 : x_3 : x_4)$, we can assume that the equation of S is

$$x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 = 0,$$

In particular S contains the edges of the tetrahedron

$$T = \{x_1x_2x_3x_4 = 0\}$$

and Sing $S = \{v_1 \dots v_4\}$, where $v_i, i = 1 \dots 4$, is a vertex of T. Let

$$\sigma : S' \to S$$

be the blowing up of Sing S. We fix our notations as follows:

(1) $E := \sum E_{ij}$, *E_{ij} being the strict transform of* $\overline{v_i v_j}$, $1 \leq i < j \leq 4$,
(2) $F := \sum F_i$, *where* $F_i := \sigma^{-1}(v_i)$,
(3) $H \in |\sigma^* \mathcal{O}_S(1)|$.

We have $\sigma^* T \sim 2E + 3F \sim 4H$ so that $F \sim 4H - 2E - 2F$. Therefore $\mathcal{O}_{S'}(F)$ is divisible by 2 in Pic S' and there exists a double covering

$$p' : \tilde{S}' \to S'$$

branched on F. In particular we have the commutative diagram

$$\begin{tikzcd} \tilde{S}' \arrow[r, "p'"] \arrow[d, "\sigma'"] & S' \arrow[d, "\sigma"] \\ \tilde{S} \arrow[r, "p"] & S \end{tikzcd}$$

where p is branched on Sing S. We introduce now some further preliminaries and then we outline the proof of the rationality of $\mathcal{R}_4$, as it is given in [Ca].

A smooth general $C \in |2H|$ is the pull-back of a general quadratic section of S. In particular T defines on C the effective divisors of degree two:

$$e_{ij} = C \cdot E_{ij}$$

and the line bundle $L := \mathcal{O}_C(D)$, where

$$D := 2H - E + F \sim \frac{1}{2} F.$$

Lemma 4.9. *L is a non trivial 2-torsion element of* $\mathrm{Pic}^0(C)$*:*

Proof. We have $L^{\otimes 2} \cong \mathcal{O}_C(F)$. The latter is $\mathcal{O}_C$ because F is effective and $F \cap C$ is empty. To show that L is not trivial consider $\tilde{C} := p'^* C$ and observe that $p'/\tilde{C} : \tilde{C} \to C$ is induced by L. If L is trivial then we have $\tilde{C} = C_1 + C_2$, where $p'/C_i : C_i \to C$ is biregular and $C_1 \cap C_2 = \emptyset$. But then $C_1^2 C_2^2 - C_1 C_2 = 144 > 0$: against Hodge index theorem. □

As in the previous section we consider now the involution

$$t_L : \mathrm{Pic}^2(C) \to \mathrm{Pic}^2(C),$$

induced by the tensor product with L, and the intersection

$$Z = W_2^0(C) \cap t_L^* W_L^0(C).$$

The self intersection of $W_2^0(C)$ is six, hence we can expect that Z consists of 6 points. Assume that $C \in | \, 2H \, |$ is general, then this is true:

Lemma 4.10. $Z = \{\mathcal{O}_C(e_{ij}),\ 1 \leq i < j \leq 4\}$. *Moreover one has*

$$\mathcal{O}_C(e_{12} - e_{34}) \cong \mathcal{O}_C(e_{13} - e_{24}) \cong \mathcal{O}_C(e_{14} - e_{23}) \cong L.$$

Proof. We can write $2H - E = (2E_{12} + E_{13} + E_{14} + E_{23} + E_{24}) - E = E_{12} - E_{34}$. We know that $L \cong \mathcal{O}_C(2H - E)$, hence $L \cong \mathcal{O}_C(e_{12} - e_{34})$. The same argument works for the other e_{ij}'s. It remains to show that Z is finite: since C is not hyperelliptic the map $u : C \to \mathbf{P}^2$, defined by $\omega_C \otimes L$, is a

morphism, [CD] 0.6.1. Moreover u is not injective at each $f \in Z$, see [CD] 0.6.5. But then $u(C)$ is a plane cubic and C is bielliptic. This is impossible, since $|2H|$ dominates $\mathcal{M}_4$ and C is general. □

Now observe that $|H|$ is the anticanonical system of S'. In particular $|H|$ is invariant by the automorphism group $Aut\ S'$. It follows that $Aut\ S'$ is isomorphic to the symmetric group S_4 of all projectivities fixing the point $(1:1:1:1)$ and the set Sing S of the vertices of the tetrahedron T. In particular S_4 acts on $|2H|$ via this isomorphism.

Lemma 4.11. *Let C, C' be general in $|2H|$. Then the Prym pairs $(C, \mathcal{O}_C(D))$ and $(C', \mathcal{O}_{C'}(D))$ are isomorphic if and only if $C' = a(C)$ for some $a \in S_4$.*

Proof. Assume the two pairs are isomorphic. Since C and C' are canonically embedded, there exists a projective automorphism a such that $a(C) = C'$ and $a^*\mathcal{O}_C(D) \cong \mathcal{O}_{C'}$. Then, by the previous lemma, it follows that $a(T) = T$. Hence $S'' = a(S)$ is a cubic through T and contains $C' \cup \mathrm{Sing}\ T$. Then $S'' = S$ for degree reasons and $a \in Aut\ S'$. The converse is obvious. □

Theorem 4.12.
(1) *$|2H|/S_4$ is birational to $\mathcal{R}_4$,*
(2) *$|2H|/S_4$ is rational.*

Proof. (1) By the previous lemma there exists a generically injective rational map $\phi : |2H|/S_4 \to \mathcal{R}_4$ sending C to the moduli point of $(C, \mathcal{O}_C(D))$. Since $\mathcal{R}_4$ and $|2H|$ are irreducible of the same dimension, (1) follows.

(2) Let l_1, l_2, l_3 be independent linear forms on $\mathbf{P}^2$ and let $l_4 = l_1 + l_2 + l_3$. Then S' is the image of the rational map $\phi : \mathbf{P}^2 \to \mathbf{P}^3$ defined by the linear system of cubics passing through the nodes of the quadrilateral $l_1l_2l_3l_4 = 0$, [D]8.2. The strict transform of $|2H|$ by ϕ is the 9-dimensional linear system

$$l_1l_2l_3l_4q + z_1(l_1l_2l_3)^2 + z_2(l_1l_2l_4)^2 + z_3(l_1l_3l_4)^2 + z_4(l_2l_3l_4)^2 = 0,$$

where q is any quadratic form in l_1, l_2, l_3. It is the linear system of all sextics which are singular at these six nodes. S_4 acts on it just by permutations of $\{l_1\ l_2\ l_3\ l_4\}$. In [Ca], using this description of $|2H|$, the field of invariants of $|2H|$ is determined and its rationality is proved. □

The theorem implies the rationality of $\mathcal{R}_4$. Note that in this case a *unique* linear system of curves, on a fixed suitable surface, dominates the moduli space we are considering. We pass now to some unirationality questions.

4.5. **Unirationality of $\mathcal{R}_6$ via Enriques surfaces.** The unirationality of $\mathcal{R}_6$ was first proved by Donagi, see [Do1]. This proof is related to nets of quadrics and conic bundles, as we will see later.

Previously we want still to play with étale or quasi étale double coverings of surfaces, in particular with Enriques surfaces and their K3 covers. Therefore we partially reproduce a second proof given in [Ve3]. This proof relies on Enriques surfaces. We describe it, with some variations, through various steps. Some of them are just preliminaries on Enriques surfaces.

Step 0: *Enriques surfaces and $\mathcal{R}_g$*

The canonical bundle ω_S of an Enriques surface S defines a non trivial étale double covering $p : \tilde{S} \to S$ such that $\tilde{S}$ is a K3 surface.

Let $C \subset S$ be any smooth, connected curve of genus $g \geq 2$. It is well known that then $L = \omega_S \otimes \mathcal{O}_C$ is a non trivial 2-torsion element of $\mathrm{Pic}^0_2(C)$. Notice also that $\dim |C| = g - 1$, cfr. [CD] ch. I.

We recall that any moduli space of polarized Enriques surfaces $(S, \mathcal{O}_S(C))$ is 10 -dimensional. A naif count of parameters suggests that the moduli space of pairs (S, C) dominates $\mathcal{R}_g$ for $g \leq 6$. In the sequel we prove that an irreducible component of it, containing the open set of pairs (S, C) such that C is very ample, is indeed unirational and dominates $\mathcal{R}_g$ for $g = 6$.

Step 1: *Enriques and Fano polarizations*

At first we consider polarized Enriques surfaces $(S, \mathcal{O}_S(H))$, where H is an integral curve of genus 4. These are related again to the tetrahedron

$$T \subset \{x_1x_2x_3x_4 = 0\} \subset \mathbf{P}^3.$$

Let S' be a general sextic passing doubly through the edges of T. Then the normalization of S' is a general Enriques surface S and the pull-back of $\mathcal{O}_{S'}(1)$ on S is a polarization $\mathcal{O}_S(H)$ of genus 4, see [CD] 4 E.

We will say that S' is an *Enriques sextic.* Writing explicitely its equation, the linear system of Enriques sextics passing doubly through T is

$$qx_1x_2x_3x_4 + a_1(x_2x_3x_4)^2 + a_2(x_1x_3x_4)^2 + a_3(x_1x_2x_4)^2 + a_4(x_1x_2x_3)^2 = 0,$$

where q is a quadratic form and a_1, a_2, a_3, a_4 are constants.

Definition 4.6. $\mathbb{E}$ *is the previous linear system of sextic surfaces.*

$\mathbb{E}$ dominates the moduli of polarized Enriques surfaces $(S, \mathcal{O}_S(H))$, which is therefore unirational.

Definition 4.7. *A Fano polarization on an Enriques surface S is a line bundle $\mathcal{O}_S(C)$ defined by a smooth, integral curve C of genus 6.*

We recall that an Enriques surface S is said to be *nodal* if S contains an *effective* divisor R such that $R^2 = -2$. A general Enriques surface is not nodal. Nodal Enriques surfaces contain a copy of $\mathbf{P}^1$. Let $(S, \mathcal{O}_S(C))$ be a Fano polarization. For simplicity we will assume that:

- *S is not nodal,*
- *$\mathcal{O}_S(C)$ is very ample.*

A general Enriques surface S has finitely many Fano polarizations $\mathcal{O}_S(C)$ modulo automorphisms. $|C|$ defines an embedding

$$S \subset \mathbf{P}^5$$

with the properties summarized in the next theorem, see [CD] 4 H and [CV].

Theorem 4.13.
(1) *S contains exactly 20 plane cubics, coming in pairs (E_n, E'_n) such that*

$$\mathcal{O}_S(E_n - E'_n) \cong \omega_S,\ n = 1 \dots 10.$$

(2) *Every trisecant line to S is trisecant to one of them.*

Furthermore let $|H| = |E_i + E_j + E_k|$ with $1 \le i < j < k \le 10$. Then $|H|$ defines a generically injective morphism $f : S \to \mathbf{P}^3$ with the properties listed below, see [CD] 4 H.

Theorem 4.14. *Let*

$$S' := f(S) \subset \mathbf{P}^3,$$

up to projectivities we have:

(1) *$f : S \to S'$ is finite,*
(2) *S' is an Enriques sextic and belongs to $\mathbb{E}$,*
(3) *$f(E_n)$, $f(E'_n)$ are skew edges of T, $n = i, j, k$.*

The same properties hold replacing E_n by E'_n for some $n = i, j, k$. Let $C \in |\mathcal{O}_S(1)|$ be a hyperplane section of $S \subset \mathbf{P}^5$. Considering $f(C)$ we have:

Lemma 4.15.
(1) *The only element of $\mathbb{E}$ containing $f(C)$ is S'.*
(2) *$\mathcal{O}_C(H)$ is non special so that $h^0(\mathcal{O}_C(H)) = 4$.*
(3) *$\mathcal{O}_C(H)$ is very ample for a general C.*

Proof. (1) Assume $f(C) \subset S''$, where $S'' \in \mathbb{E}$ and $S'' \neq S'$. Then f^*S'' is the curve $4\sum(E_n + E'_n) + C + R \in |6H|$, with $R \in |2H - C|$ and $R^2 = -2$. This is a contradiction because we assume that S is non nodal.

(2) Since $(H - C)^2 = -2$ and S is not nodal, it follows from Riemann-Roch that $h^i(\mathcal{O}_S(H - C)) = 0, i = 0, 1, 2$. Then the non speciality of $\mathcal{O}_C(H)$ follows from the standard exact sequence

$$0 \to \mathcal{O}_S(H - C) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0.$$

(3) C is very ample and $f : S \to S'$ is a finite, generically injective morphism. Then, counting dimensions, the locus of all $D \in |C|$ such that f/D is not an embedding is a proper closed subset. $\square$

Curves of genus 6 and degree 9 on Enriques sextics

Keeping the previous notations, we want to study the embedded curve

$$C' := f(C) \subset \mathbf{P}^3.$$

At first we fix some further conventions: we denote by L_m and L'_m the skew edges of T such that

$$f^*L_m = E_m \ , \ f^*L'_m = E'_m, \quad m = i, j, k.$$

We also assume that L_i, L_j, L_k are *coplanar* lines. It is clear that

Proposition 4.16. *$f^*\mathcal{O}_{C'}(1) \cong \mathcal{O}_C(e_i + e_j + e_k)$, so that $(f/C)^*(L_m) = e_m$ and $(f/C)^*(L'_m) = e'_m$. In particular each edge of T is trisecant to C'.*

Finally we put

$$e_n = E_n \cdot C \ , \ e'_n = E'_n \cdot C, \ n = 1 \dots 10.$$

Since $E_n - E'_n \sim E'_n - E_n$ is a canonical divisor on S, the line bundle

$$L := \mathcal{O}_C(e_n - e'_n)$$

is a non trivial 2-torsion element of $\mathrm{Pic}^0(C)$. The tensor product by L defines a translation $t_L : \ \mathrm{Pic}^3(C) \to \ \mathrm{Pic}^3(C)$. For a general C we consider $W^0_3(C) \subset \mathrm{Pic}^3(C)$ and, as previously, the intersection

$$Z := W^0_3(C) \cap t^*_L W^0_3(C).$$

Proposition 4.17. $Z = \{e_n, \ e'_n, \ n = 1 \dots 10\}$ *and the intersection is transversal.*

Proof. It is obvious that $e_n, e'_n \in Z$. Since the self intersection of $W^0_3(C)$ is 20, the statement follows if Z is finite. To prove this recall that $C \subset S \subset \mathbf{P}^5$ and that $\mathcal{O}_C(1) \cong \omega_C \otimes L$. It is standard to check that $e \in Z$ if and only if $L \cong \mathcal{O}_C(e - e')$ with e' effective if and only if e is contained in a trisecant line to C. But it is shown in [CV] that every trisecant line to a smooth, non nodal Fano model $S \subset \mathbf{P}^5$ is contained in the plane spanned by one cubic E_n or E'_n. Hence e is e_n or e'_n, for some $n = 1 \dots 10$, and Z is finite. □

Definition 4.8. $\mathbb{F}$ *is the family of all curves* $C' \subset \mathbf{P}^3$ *such that*

(1) C' *is smooth, connected of degree nine and genus six,*
(2) C' *is contained in a not nodal Enriques sextic* $S' = f(S) \in \mathbb{E}$*,*
(3) $C' = f(C)$ *where* $\mathcal{O}_S(C)$ *is a Fano polarization.*
(4) $Cf^*L_m = Cf^*L'_m = 3$*,* $m = i, j, k$*.*

We point out that condition (2) implies that the Enriques sextic $S' \in \mathbb{E}$ is *unique*, this is due to lemma 4.14. We recall also that $f : S \to S'$ is just the normalization map.

Definition 4.9. $\tilde{\mathbb{E}}$ *is the family of pairs* $(S', \mathcal{O}_S(C))$ *such that*

(1) $S' \in \mathbb{E}$ *is a non nodal Enriques sextic,*
(2) $f : S \to S'$ *is the normalization map,*
(3) $C' := f(C)$ *belongs to* $\mathbb{F}$*.*

Since $C' \in \mathbb{F}$ uniquely defines S', we have a morphism

$$p : \mathbb{F} \to \tilde{\mathbb{E}}$$

sending C' to the pair $(S', \mathcal{O}_S(C))$. The fibre of p at $(S', \mathcal{O}_S(C))$ is a non empty open set of the 5-dimensional linear system $|C|$.

It is standard to construct the family $q : \mathcal{S} \to \tilde{\mathbb{E}}$ which is the normalization of the universal Enriques sextic $\mathcal{S}' \subset \tilde{\mathbb{E}} \times \mathbf{P}^3$. It is standard as well to construct a line bundle $\mathcal{C}$ over $\tilde{\mathbb{E}}$ whose restriction to the fibre S of q at $x := (S', \mathcal{O}_S(C))$ is $\mathcal{O}_S(C)$. Then Grauert theorem implies that, on a dense open set of $\tilde{\mathbb{E}}$, $q_*\mathcal{C}$ is a vector bundle with fibre $H^0(\mathcal{O}_S(C))$ at x. $\mathbf{P}\mathcal{C}$ is clearly birational to $\mathbb{F}$, hence it follows that:

Lemma 4.18. *$\mathbb{F}$ is birational to $\mathbf{P}^5 \times \tilde{\mathbb{E}}$.*

On the other hand the assignement $C' \mapsto (C, L)$ defines the moduli map

$$m : \mathbb{F} \to \mathcal{R}_6,$$

where C and $L \cong \mathcal{O}_C(e_n - e'_n)$ are obtained as above from C'.

Proposition 4.19. *m is dominant.*

Proof. Let $C' \in \mathbb{F}$ and let (C, L) be in the isomorphism class of $m(C')$. Then , keeping the previous notations, C' is embedded in $\mathbf{P}^3$ by a line bundle $\mathcal{O}_C(a_i + a_j + a_k)$ for some a_i, a_j, a_k in $Z := W^0_3(C) \cap t^*_L W^0_3(C)$. By proposition 4.16 Z is finite. Moreover we know that $h^0(\mathcal{O}_{C'}(1)) = 4$ by lemma 4.14. Let $C'' \in \mathbb{F}$, then $m(C'') = m(C')$ if and only if C'' is biregular to C' and $\mathcal{O}_{C''}(1) \cong \mathcal{O}_{C'}(b_i + b_j + b_k)$ for some $b_i, b_j, b_k \in Z$. In particular C'' and C' are projectively equivalent if and only if $C'' = \alpha(C')$ for some $\alpha \in Aut\ T$, the group of projectivities of T. Since $Aut\ T$ is 3-dimensional and Z is finite, it follows that the fibre of m over $m(C')$ is 3-dimensional. Hence m is dominant if and only if $\dim\ \mathbb{F} \geq \dim\ \mathcal{R}_6 + \dim\ Aut\ T = 18$. To prove this note that the natural projection $u : \tilde{\mathbb{E}} \to \mathbb{E}$ has finite fibres. Indeed the fibre of u at S' is the set of Fano polarizations of S. Hence $\dim\ \tilde{\mathbb{F}} = \dim\ \mathbb{F} = 13$ and $\dim\ \mathbb{F} = 18$. □

Theorem 4.20. *$\tilde{\mathbb{E}}$ is irreducible and rational.*

This theorem is proven in [Ve3]. Since $\tilde{\mathbb{E}}$ is rational then $\mathbb{F}$ is rational too. Since $m : \mathbb{F} \to \mathcal{R}_6$ is dominant it follows:

Corollary 4.21. *$\mathcal{R}_6$ is unirational.*

4.6. **The unirationality of $\mathcal{R}_g$ via conic bundles: $g \leq 6$.** We continue along the path of the unirationality of $\mathcal{R}_g$ adding more facts and a few historical remarks. Then we will discuss some well known relations between conic bundles and Prym pairs. Finally we will complete our picture giving a simple proof via conic bundles of the following known result

- ◦ *$\mathcal{R}_g$ is unirational for $g \leq 6$.*

The proof uses linear systems of nodal conic bundles in $\mathbf{P}^2 \times \mathbf{P}^2$. So it follows, in some sense, the spirit of classical parametrizations of $\mathcal{M}_g$ via families of nodal plane curves, we started with in the first part of this exposition.

A brief historical overview

The first proof of the unirationality of $\mathcal{R}_6$, given by Donagi in [Do1], goes back to 1984. A further purpose of this paper is to prove the unirationality of $\mathcal{A}_5$, the moduli space of principally polarized abelian varieties. The author considers Fano threefolds, of index one and of the principal series,

$$V \subset \mathbf{P}^7$$

which are a complete intersection of type $(1,1,2)$ in the Grassmannian $G(2,5)$. Among them one has the family of those threefolds having five ordinary double points. The purpose is to show that the intermediate Jacobian of a minimal desingularization of a general threefold in this family is in fact a general 5-dimensional principally polarized abelian variety. As we will see, this is actually the case.

Projecting from one of the nodes of V, one obtains a complete intersection of three quadrics in $\mathbf{P}^6$. Actually this net of quadrics is quite special. The discriminant curve of the net is a plane curve $\Gamma \cup P$, where P is a line and Γ is a 4-nodal sextic. It turns out that the normalization C of Γ is endowed with a non split étale 2:1 cover

$$\pi : \tilde{C} \to C.$$

Γ parametrizes a family of singular quadrics $Q_x, x \in \Gamma$, of the net. $\tilde{C}$ is the normalization of the family of rulings of maximal linear subspaces of the quadrics Q_x, $x \in \Gamma$. An important fact is that every such a Q_x has rank 6, even when x is singular for $\Gamma \cup P$. This implies that π is étale.

Let L be the non trivial line bundle defining π. Then (C, L) is a Prym pair. The family of 5-nodal Fano threefolds V is easily seen to be rational. Hence, for proving in this way the unirationality of $\mathcal{R}_6$ and of $\mathcal{A}_5$, the crucial point is to show that the previous Prym pais (C, L) is general in moduli. This step is done in [Do1] via an appropriate algorithm. We will see in a moment a simple variation of this method.

Before we want to complete our picture with a brief report on the unirationality of $\mathcal{R}_5$. The first complete proof of the unirationality of $\mathcal{R}_5$ is quite new. This result is due to Izadi, Lo Giudice and Sankaran, [ILS]. An independent proof follows from [Ve4].

All the subject, as well as some of the proofs, was strongly influenced by the study of rational parametrizations for the moduli spaces $\mathcal{A}_g$ of principally polarized abelian varieties of dimesion $g \leq 6$. In [C] Clemens was the first to produce a rational parametrization of the moduli space $\mathcal{A}_4$. He studies a special family of threefods, namely quartic double solids

$$f : V \to \mathbf{P}^3,$$

branched on a quartic surface B with 6 nodes. The method is the same used by Donagi for $\mathcal{A}_5$: the intermediate Jacobian of the blowing up of Sing V is 4-dimensional. Then the main point is to show that the period map

$$p : \mathcal{V} \to \mathcal{A}_4$$

is dominant. Also in this case Prym pairs are present in the construction, though not considered in the proof. Projecting from a node of B one can realize a birational model of V as a singular conic bundle over $\mathbf{P}^2$. In this case the discriminant $\Gamma \subset \mathbf{P}^2$ is a sextic with 5 nodes. Its normalization C is endowed with an étale double cover $\pi : \tilde{C} \to C$, parametrizing the irreducible components of the singular fibres of this conic bundle.

Families of nodal conic bundles on $\mathbf{P}^2$ *and* $\mathcal{R}_g$

We want to see, more in general, some useful families of nodal conic bundles. At first we define the conic bundles on $\mathbf{P}^2$ we want. Consider a rank 3 vector bundle E over $\mathbf{P}^2$ and the natural projection $p : \mathbf{P}E \to \mathbf{P}^2$. Let

$$V \subset \mathbf{P}E$$

be a hypersurface, we introduce the following

Definition 4.10. *V is an admissible conic bundle of $\mathbf{P}E$ if:*

(1) *$p/V : V \to \mathbf{P}^2$ is flat,*
(2) *each fibre of p/V is a conic of rank ≥ 2,*
(3) Sing V *is a finite set of δ nodes,*
(4) *the union of the singular fibres of p is an integral surface.*

By definition the discriminant curve of V is

$$C' := \{x \in \mathbf{P}^2 \ / \ rk\ p^*(x) = 2\}.$$

To give a δ-nodal conic bundle $V \subset \mathbf{P}E$ is equivalent to give

$$q : E \otimes E \to \mathcal{O}_{\mathbf{P}^2}(m),$$

where q is a quadratic form on E defining the exact sequence

$$0 \to E \to E^*(m) \to A \to 0.$$

Furthermore conditions (2) and (4) imply that

- *A is a line bundle on a C',*
- *C' is an integral δ-nodal curve,*

It is easy to see that the map $p/\operatorname{Sing} V : \operatorname{Sing} V \to \operatorname{Sing} C'$ is bijective. Moreover C' is endowed with the étale double covering

$$\pi' : \tilde{C}' \to C',$$

where

$$\tilde{C}' = \{l \in \mathbf{P}E^*_x \ / \ l \subset p^*(x),\ x \in C'\}$$

is the family of the irreducible components of the singular fibres of p/V. $\tilde{C}'$ is integral by condition (4). We denote the normalization map of C as

$$\nu : C \to C'.$$

Clearly π' is defined by a non trivial 2-torsion element

$$L' \in \operatorname{Pic}^0(C').$$

Moreover we have the standard exact sequence

$$0 \longrightarrow \mathbf{C}^{*\delta} \longrightarrow \operatorname{Pic}^0(C') \xrightarrow{\nu^*} \operatorname{Pic}^0(C) \longrightarrow 0.$$

Hence L' defines a line bundle

$$L := \nu^* L'$$

such that $L^{\otimes 2} \cong \mathcal{O}_{\tilde{C}}$.

Lemma 4.22. *L is non trivial.*

Proof. Consider the cartesian square

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\pi} & C \\ \Big\downarrow{\scriptstyle \tilde{\nu}} & & \Big\downarrow{\scriptstyle \nu} \\ \tilde{C}' & \xrightarrow{\pi'} & C', \end{array}$$

where $\tilde{\nu}$ is the normalization map of $\tilde{C}'$ and π is defined by L. Since $\tilde{C}'$ is integral, $\tilde{C}$ is integral too. Hence π is non split and L is not trivial. □

Definition 4.11. *(C, L) is the Prym pair of the conic bundle V.*

We will assume that C has genus g. From now on we denote as

$$|V|_\delta$$

the family of admissible conic bundles $W \in |V|$ having exactly δ nodes.

Definition 4.12. *The natural map is the morphism*

$$m : |V|_\delta \to \mathcal{R}_g$$

sending an admissible conic bundle $V' \in |V|$ to the moduli of its Prym pair.

It is interesting to study m in the simplest cases of E. For instance if E is the direct sum of three line bundles. Let $\mathcal{O}_{\mathbf{P}E}(1)$ be the tautological bundle. The linear system $|V|$ we want to use is $|\mathcal{O}_{\mathbf{P}E}(2)|$. The case

$$E = \mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 3}$$

is already quite interesting. Indeed it provides a quick proof of the unirationality of $\mathcal{R}_g$, $g \leq 6$. In this case we have

$$|V| = \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2, 2)|.$$

Fixing coordinates $(x, y) := (x_1 : x_2 : x_3) \times (y_1 : y_2 : y_3)$, it follows that

$$|V| = \{ \sum_{1 \leq i < j \leq 3} a_{ij} y_i y_j = 0 \},$$

where $M := (a_{ij})$ is a 3×3 symmetric matrix of quadratic forms in x. Let V be standard with discriminant curve C' and let $L' \in \mathrm{Pic}^0(C')$ be the non trivial line bundle considered above. Then M defines the exact sequence

$$0 \longrightarrow \mathcal{O}^3_{\mathbf{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbf{P}^2}(-1)^3 \longrightarrow L' \longrightarrow 0.$$

Note that C' is an integral sextic with δ nodes and that V provides a sextic equation of C' as the determinant of a symmetric matrix of quadratic forms. It is well known that to give such a determinant is equivalent to give a non trivial element of $\mathrm{Pic}^0_2(C')$ as L', cfr. [Be1] lemma 6.8 and also [DI] 6.1.

Lemma 4.23. *The natural map $|V|_\delta \to \mathcal{R}_g$ is dominant for $g \leq 6$.*

Proof. Let (C, L) be a general Prym pair, with C of genus $g \leq 6$. Then C is birational to a plane sextic C' with $\delta = 10 - g$ nodes. Let $L' \in \nu^{*-1}(L)$, where $\nu : C \to C'$ is the normalization map. Then L' defines an exact sequence as above, hence a matrix M as above. Consider the conic bundle V, defined by the equation $xMy = 0$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Since L' is a line bundle, it easily follows that V is admissible and in $|V|_\delta$. Hence $m_g(V)$ is the moduli point of (C, L) and the statement follows. □

Like in the case of nodal plane curves we can now consider the family of δ-nodal conic bundles V. Since dim $|V| = 35$, we can assume that the points of Sing V are in general position for $4 \leq \delta \leq 7$. Then, as in the case of nodal plane curves, $|V|_\delta$ is rational. This implies that:

Theorem 4.24. *$\mathcal{R}_g$ is unirational for $g \leq 6$.*

Remark 4.1. For $g = 6$ then Sing V consists of 4 general points. Since all these sets are projectively equivalent in $\mathbf{P}^2 \times \mathbf{P}^2$ we can fix one of them, say $Z = \{o_1, o_2, o_3, o_4\}$. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. We point out that then $\mathcal{R}_6$ *is dominated by a linear system,* namely by

$$|\mathcal{I}_Z^2(2,2)|.$$

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